Matching Theory and Market Design

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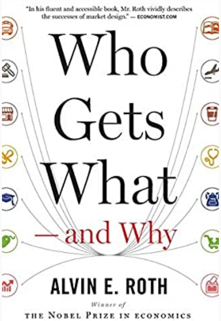
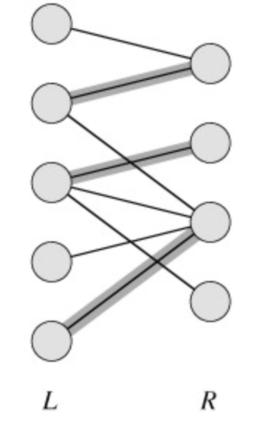
UNSW Sydney

Who gets what?

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

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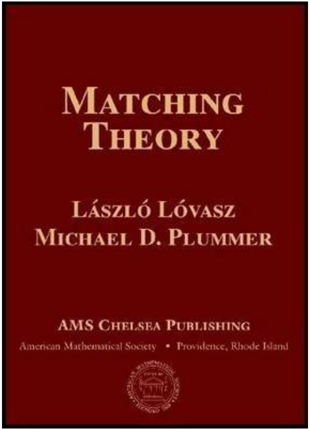
MATCHING
THEORY
LÁSZLÓ LÓVASZ
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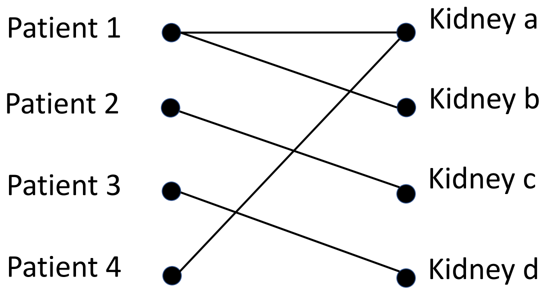
Goals

- Basic overview of matching theory and biparite graphs
- Glimpse into algorithmic market design
- Familiarize with key concepts
- Increased appreciation of the axiomatic method
- Understanding of useful market design algorithms

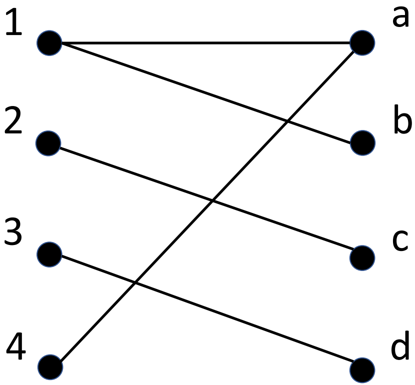
Maximal Allocation Problem

An Allocation Problem



QUESTION: Based on the compatibility relations, how can we match the patients to the kidneys?

Bipartite Graph



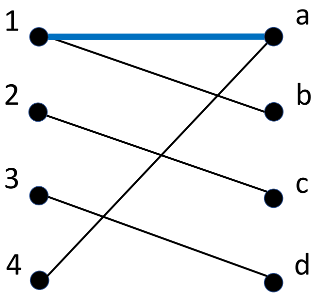
Bipartite Graph

Lemma
The following are equivalent for an undirected graph G.

1. *G is bipartite*
2. *G is 2-colorable*
3. *G does not have a cycle of odd size.*

Matching

Matching: Give an undirected graph $G = (V, E)$, a matching M is a subset of the edges $M \subseteq E$ such that each vertex $v \in V$ is incident to at most one edge from M.



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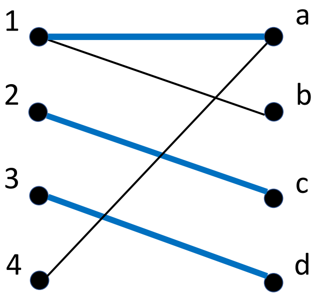

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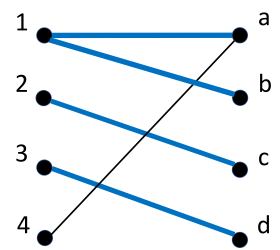

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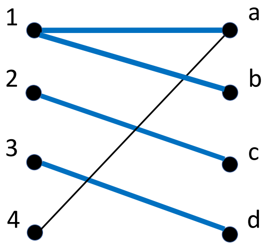
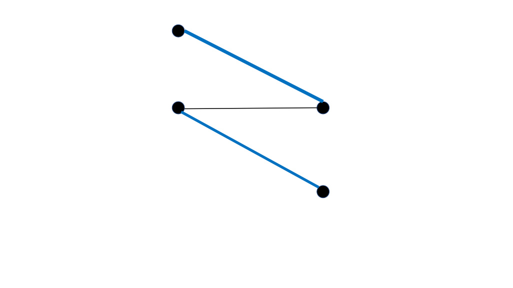


Not a matching.

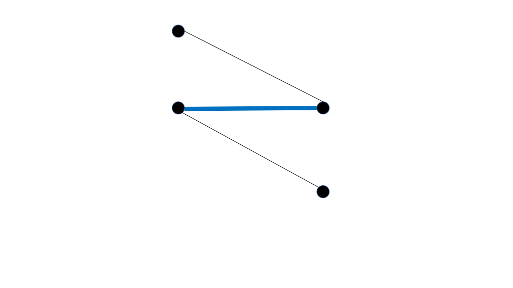
Alternating Path

- An **alternating path** with respect to a matching M is a path in which edges alternate between those in M and those not in M .
- A **matched vertex** is one incident to an edge in M
- An **free vertex** is a vertex that is not matched



Augmenting Path

- An **augmenting path** is an alternating path that starts and ends with a free vertex.



Augmenting Path

Lemma (Berge's Lemma)
A matching M is maximum size $\iff$ there is no augmenting path relative to the matching M.

Homework: Try to prove this (a nice simple exercise to practice mathematical proofs).

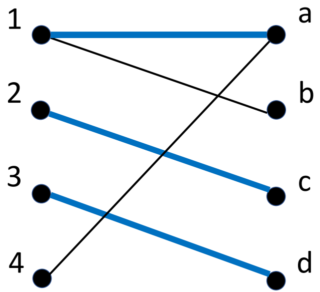
Augmenting Path

Lemma (Berge's Lemma)
A matching M is maximum size $\iff$ there is no augmenting path relative to the matching M.

Kuhn's Algorithm for Maximum Bipartite Matching

First, we take an empty matching $M = \emptyset$. While there is an augmenting path, we update M by alternating it along this path. Return M.

Finding a matching of larger size



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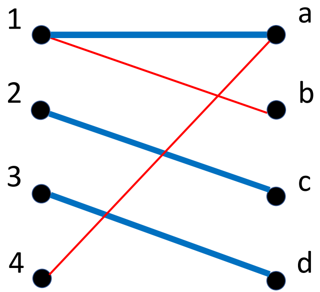
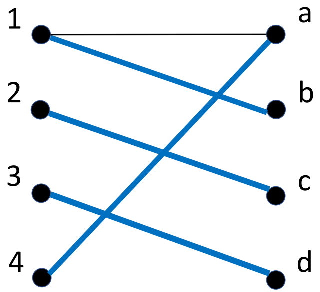
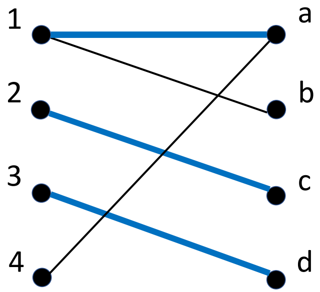


Finding a matching of larger size



How to Find an Augmenting Path



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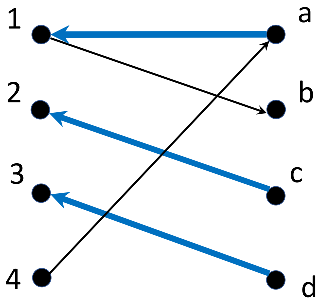

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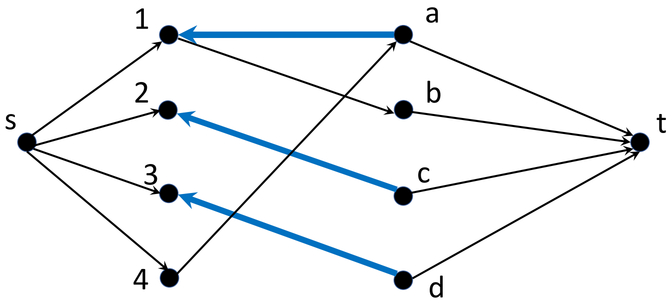

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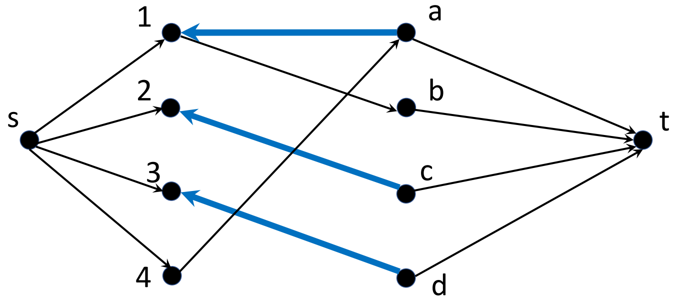


Homework: explore connections between network flows and maximum size matchings of bipartite graphs.

Augmenting Path

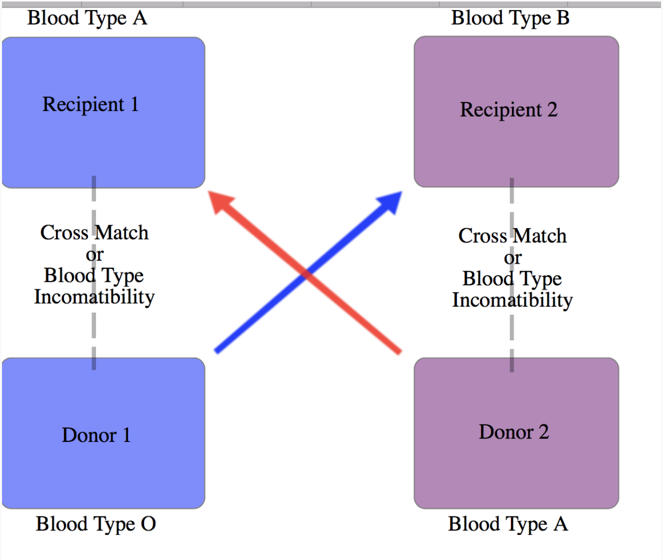
Lemma (Berge's Lemma)
A matching M is maximum size $\iff$ there is no augmenting path relative to the matching M.

Kuhn's Algorithm for Maximum Bipartite Matching

First, we take an empty matching $M = \emptyset$. While there is an augmenting path, we update M by alternating it along this path. Return M.

An Exchange Problem

Organ Markets



An exchange problem

- Each agent i owns a single item o_i
- Agents have preferences over items

The preference ranking of the agents over items are from left to right. Owned items are underlined:

$$\begin{aligned}
\succ_1&: \quad o_2, o_3, \underline{o_1}, o_4 \\
\succ_2&: \quad o_3, o_1, \underline{o_2}, o_4 \\
\succ_3&: \quad o_1, \underline{o_3}, o_4, o_2 \\
\succ_4&: \quad o_1, \underline{o_4}, o_2, o_3
\end{aligned}$$

Question: Who should get what item?

Individual Rationality

The outcome should be at least as preferred by each agent as their 'backup' outcome.

Pareto efficiency

There should be no outcome that is weakly better for everyone and strictly better for at least someone.

Core Stability

There is no coalition of agents who can deviate and obtain a unanimously better outcome by cooperating within the deviating coalition.

Strategyproofness

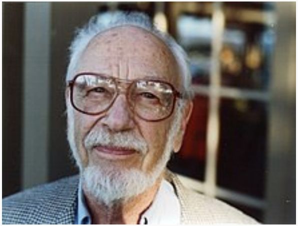
Agents never have an incentive to misreport their preferences to obtain a more preferred outcome.

- Optimisation can be done on the right input
- Levels the playing field against agents who have more information.

Gale's Top Trading Cycles Mechanism

TTC enables of exchanges of items.

Proposed by mathematician David Gale



Gale's Top Trading Cycles Mechanism

TTC is strategyproof and its outcome is **Pareto efficient** (no allocation is unanimously better for agents), **individually rational** (outcome of each agent is at least as good as the intial allocation), and **core stable**.[1]

[1]

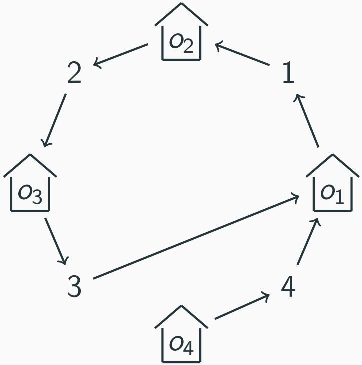
Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

$\succ_1$: o_2, o_3, o_1, o_4

$\succ_2$: o_3, o_1, o_2, o_4

$\succ_3$: o_1, o_3, o_4, o_2

$\succ_4$: o_1, o_4, o_2, o_3



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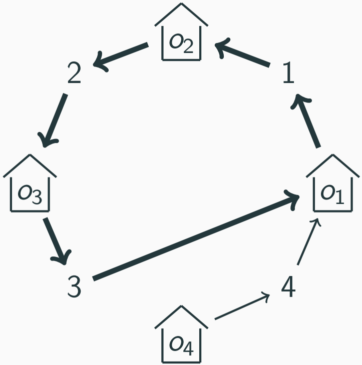

Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

$\succ_1$: o_2, o_3, o_1, o_4

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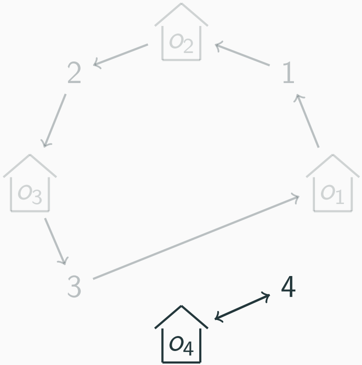
$\succ_4$: o_1, o_4, o_2, o_3



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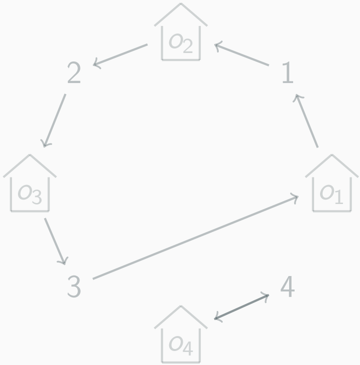
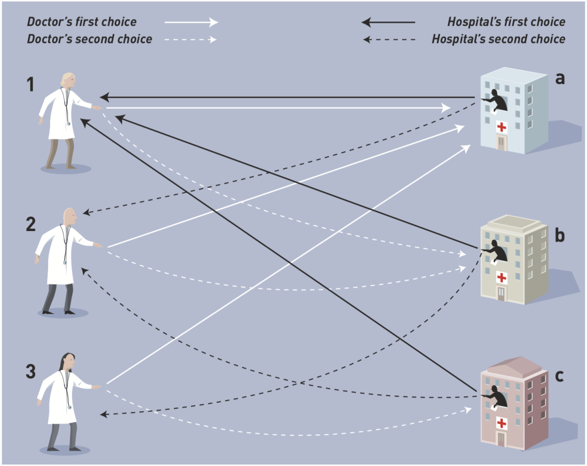


Exchange Markets: some references

Sonmez, Tayfun and M. Utku Unver (2011), "Matching, allocation, and exchange of discrete resources." In Handbook of Social Economics, volume 1B (Jess Benhabib, Matthew O. Jackson, and Alberto Bisin, eds.), 781-852, North-Holland, San Diego.

Allocation Under Priorities

Job Markets



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[2]https://www.nobelprize.org/uploads/2018/06/popular-economicsciences2012.pdf

Allocation Under Priorities

- Agents have preferences over items. Items have priorities over agents. Each agent needs one item.

Agents 1,2,3 have preferences over items a, b, c. The items have priorities over agents.

$b \succ a \succ c$	①	ⓐ	$1 \succ 3 \succ 2$
$a \succ b \succ c$	②	ⓑ	$2 \succ 1 \succ 3$
$a \succ b \succ c$	③	ⓒ	$2 \succ 1 \succ 3$

Who should get what item?

Allocation Under Priorities

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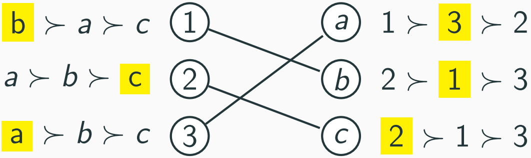
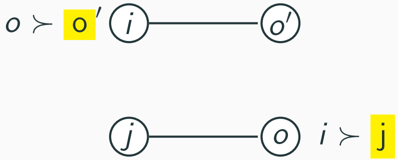


QUESTION: Is this allocation fair?

Allocation Under Priorities

Violation of justified envy-freeness:



Gale-Shapley's Deferred Acceptance Mechanism

Deferred Acceptance

- Agents from one side make 'proposals' to the other side.
- Items choose the best partner agents from among available proposals and rejects others.
- Rejected agents apply to the next most items.

Gale-Shapley's Deferred Acceptance Mechanism

The Agent Proposing Deferred Acceptance Algorithm is strategyproof and returns an outcome that satisfies justified envy-freeness and constrained Pareto efficiency.

Agent Proposing DA (Deferred Acceptance)

$b \succ a \succ c$	①	ⓐ	$1 \succ 3 \succ 2$
$a \succ b \succ c$	②	ⓑ	$2 \succ 1 \succ 3$
$a \succ b \succ c$	③	ⓒ	$2 \succ 1 \succ 3$

- 2 and 3 apply to a; 1 applies to b
- a rejects 2 in favour of 3
  $\{\{1, b\}, \{3, a\}\}$
- 2 applies to b; b rejects 1 in favour of 2
  $\{\{2, b\}, \{3, a\}\}$

Agent Proposing DA (Deferred Acceptance)

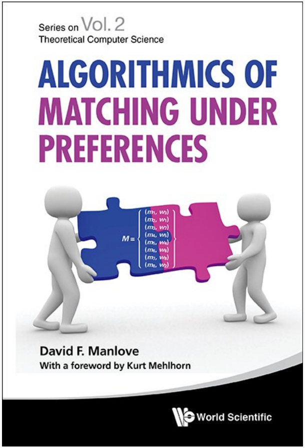
$b \succ a \succ c$	①	ⓐ	$1 \succ 3 \succ 2$
$a \succ b \succ c$	②	ⓑ	$2 \succ 1 \succ 3$
$a \succ b \succ c$	③	ⓒ	$2 \succ 1 \succ 3$

- $\{\{2, b\}, \{3, a\}\}$
  1 applies to a
- a rejects 3 in favour of 1
  $\{\{2, b\}, \{1, a\}\}$
- 3 applies to b; b rejects 3 in favour of 2
- 3 applies to c and gets accepted.
  {{1,a}{2,b},{3,c} }.

Allocation Under Priorities: Some References

- Abdulkadiroğlu, A.; and Sönmez, T. 2003. School Choice: A Mechanism Design Approach. *American Economic Review* 93(3): 729–747.
- Roth, A. E. 2008. Deferred acceptance algorithms: history, theory, practice, and open questions. International Journal of Game Theory 36:537-569.

Books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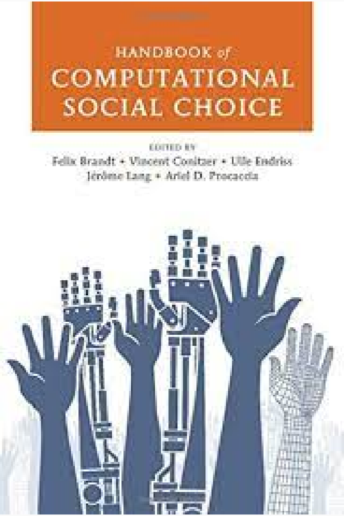
HANDBOOK of
COMPUTATIONAL
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Conclusions

- Matching Theory and graph theory in general help solve important and fundamental problems
- Algorithmic Market Design has far-reaching consequences and takes into account both axiom design and algorithm design
- We studied some key concepts (individual rationality, Pareto efficiency, core stability, strategyproofness)
- We studied some algorithms (Kuhn's Algorithm; TTC and Deferred Acceptance).