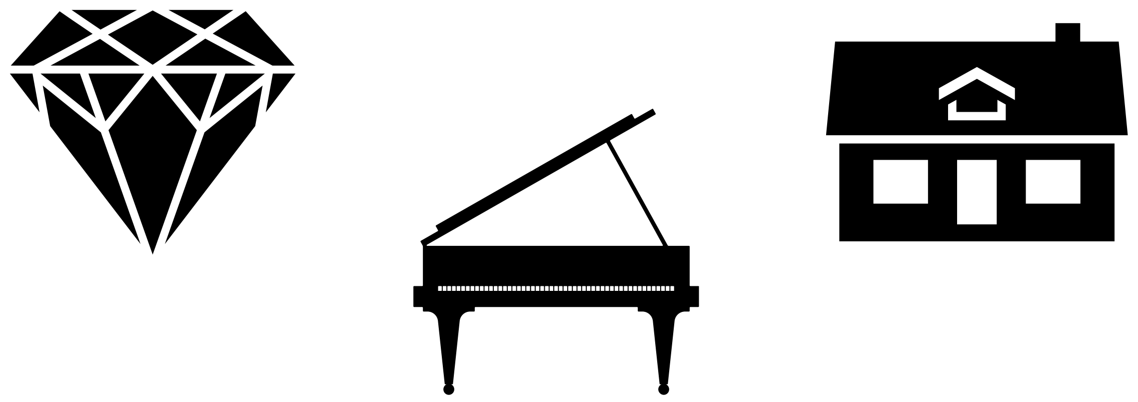
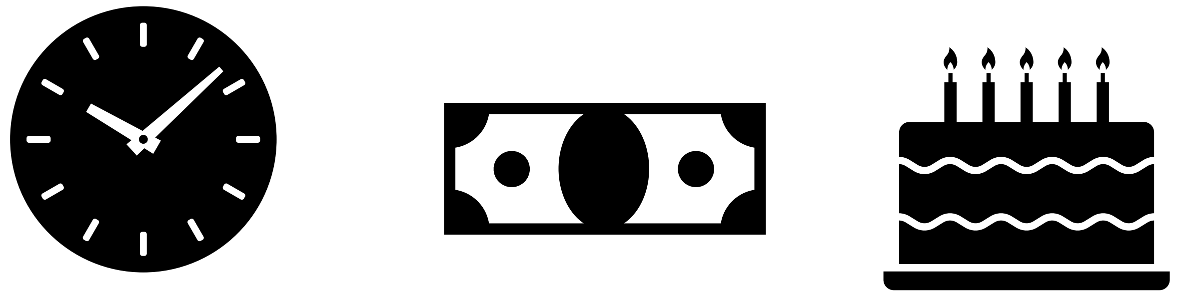
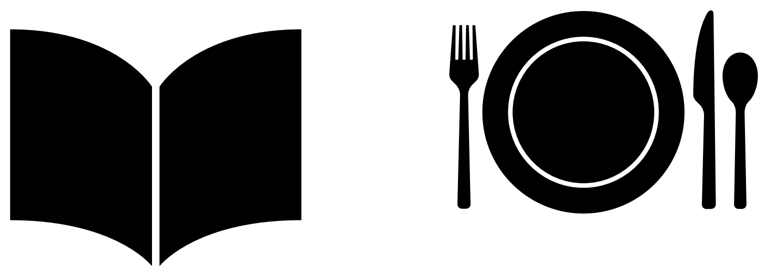
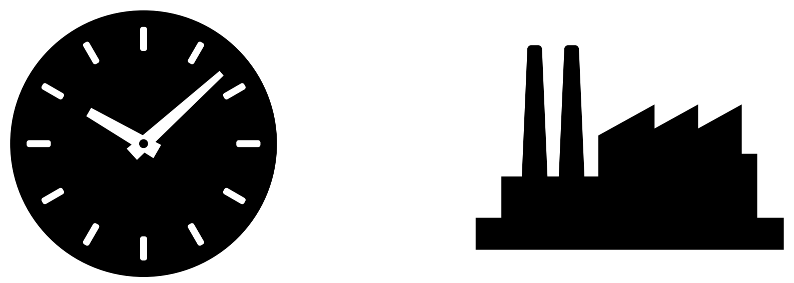
Fair Division with Subsidy

Mashbat Suzuki

AJCAI 2022

Perth, Australia

Quick overview of "Realm of Fair Division"

	Indivisible	Divisible
Goods		
Chores		

Fair Allocation of Indivisible Goods

Set of Agents	$N = \{1, 2, \ldots, n\}$
Set of Items	$M = \{1, 2, \ldots, m\}$

Fair Allocation of Indivisible Goods

Set of Agents $N = \{1,2,\ldots,n\}$

Set of Items $M = \{1,2,\ldots,m\}$

Agent Preferences over the set of items are modelled using a "valuation function"

$$u_i : 2^M \to \mathbb{R}_+$$

$u_i(S)$ Represents how much agent i value the bundle S of items

Different types of valuation functions

-Additive $u_i(S) = \sum_{j \in S} u_i(j)$

-Submodular $u_i(S \cup T) + u_i(S \cap T) \leq u_i(S) + u_i(T) \quad \forall S, T \subseteq M$

-Subadditive $u_i(S \cup T) \leq u_i(S) + u_i(T) \quad \forall S, T \subseteq M$

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Common Assumptions in Fair Division

Normalized	$u_i(\emptyset) = 0$
Monotone	$u_i(S) \leq u_i(T) \quad \forall S \subseteq T$

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Captures complementarities

Allocation $A = (A_1, \cdots, A_n)$ is a partition of the item set into n sets

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General goal = Find "fair" allocations

Quintessential Notion of Fairness

Given an allocation A, agent i **envy** agent j if $u_i(A_i) < u_i(A_j)$

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	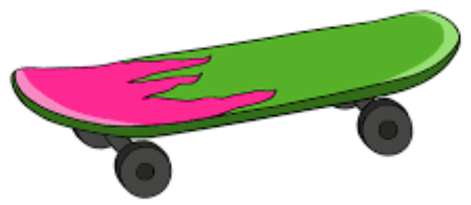		
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	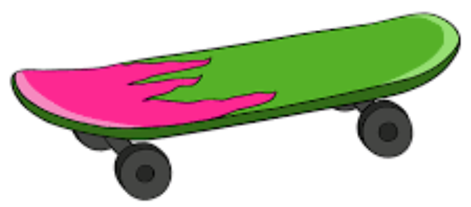		
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	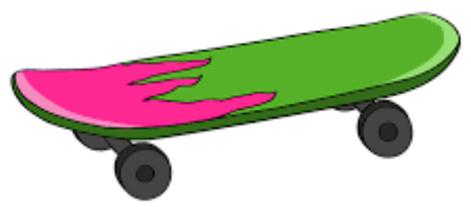		
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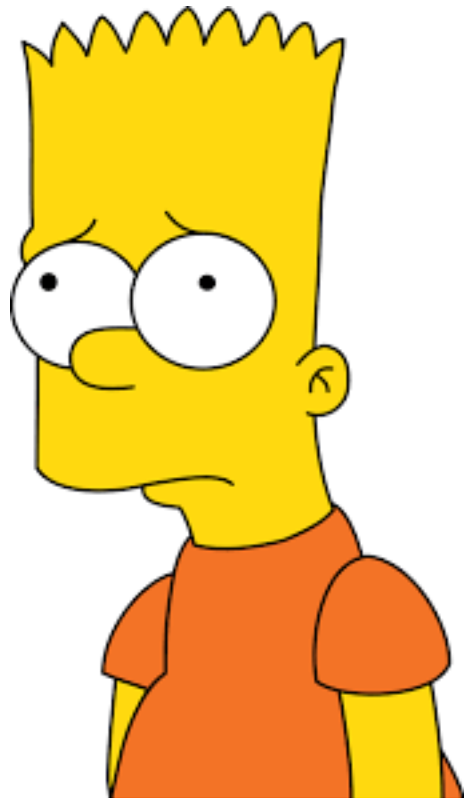
Not envy-free!



Bart envies Lisa!

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	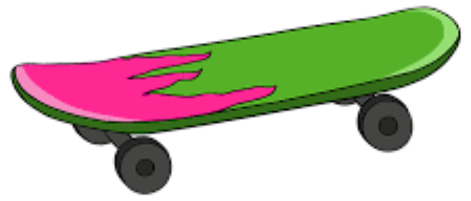		
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	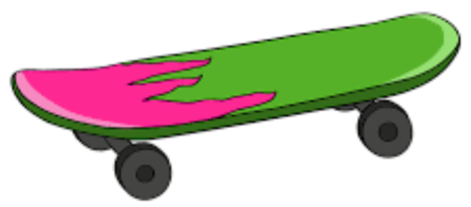		
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Lisa envies Bart!

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	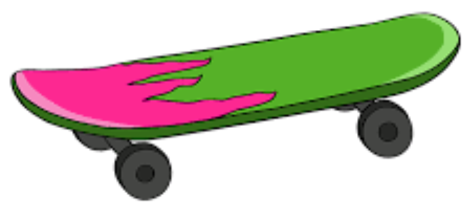		
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There is no more envy! Its an envy-free allocation

Envy-Free allocations do ***not*** always exist !

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Consider two agents and a single indivisible good!

Envy-Free allocations do ***not*** always exist !

Theorem: Checking whether there exist an EF allocation is NP-hard

Relaxations of Envy-Freeness

- An allocation A is **Envy-Free up to One Item (EF1)** if for each $i, j \in N$

$$u_i(A_i) \geq u_i(A_j \backslash g) \quad \text{for some } g \in A_j$$

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	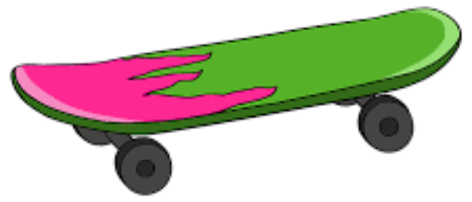		
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	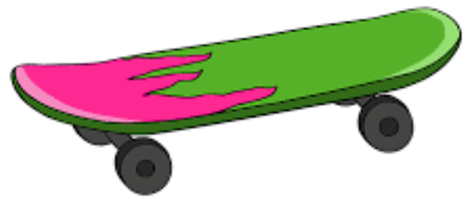		
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	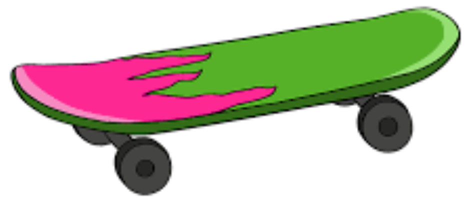		
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	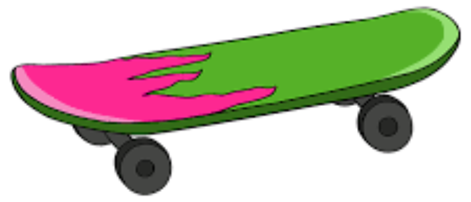		
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EFX but **NOT** EF

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EFX is too hard!

$n = 2$	$n = 3$	$n \geq 4$
You divide, I choose. Often called "Cut-n-Choose"	Very complicated existence proof!	Existence unknown! A major open problem in fair division

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Book of Genesis, Bible	"EFX Exists for Three Agents"	
1200~165 BC	2020	?

What about EF1 allocations?

Common Algorithms for EF1 Allocations

- Additive Valuations

-Round Robin

Arbitrary order the agents and let each agents pick their favourite items among the unallocated items

-Maximize Nash Social Welfare

$$\mathrm{MNW} = \max_{A} \prod_{i=1}^{n} u_i(A_i)$$

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- General Valuations

-Envy Cycle Elimination

Lipton, Markakis, Mossel, and Saberi (2004)

However EF1 allocations are often too weak!

	1	2	3	4	....	m/2	m/2 +1	....	m
	m $	1 $	1 $	1 $	....	1 $	1 $	....	1 $
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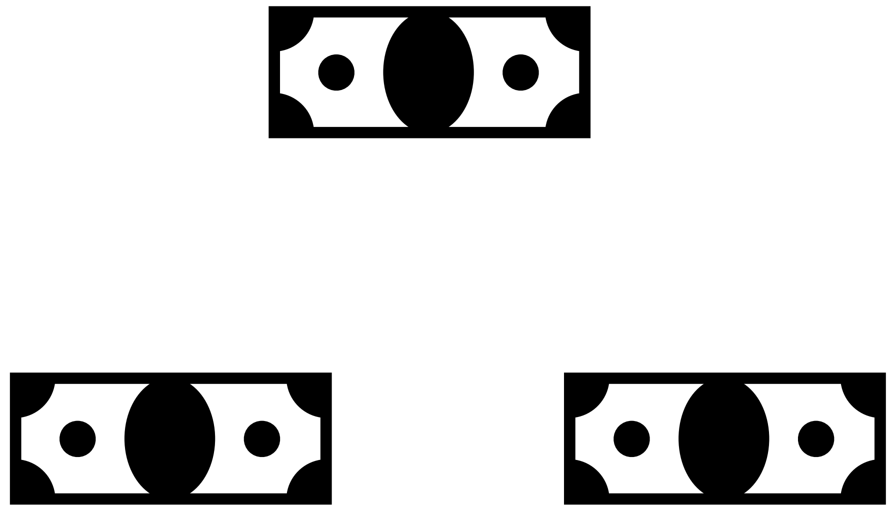
This is an EF1 allocation! But it is clearly not "fair"

Can we find EF allocation by introducing "Money" ?

Eric Maskin

2007 Nobel Prize in Economics



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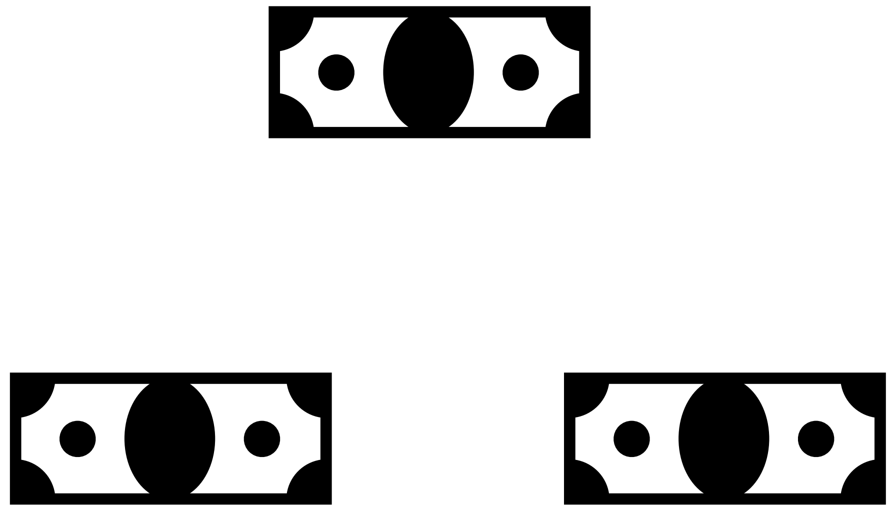

Can we find envy-free allocations by introducing "small" amounts of money?

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For simplicity we assume that the marginal value of each item is at most one dollar!
This can be acheived simply by uniformly scaling the valuation

Brief History of Fair Division with Subsidy Problem

Theorem (Maskin 86'):

In the n agent, n item, unit demand setting, envy-free allocation exists with subsidy at most $n - 1$ dollars

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Theorem (Halpern, Shah 19'):

For m-item and n-agent setting with additive valuations, envy-free allocation always exist whose subsidy is at most m(n-1)

Tight Subsidy Bounds for Additive Valuations

Theorem (Brustle, Dippel, Narayan, Suzuki, Vetta 20'):
For additive valuations, there is a polynomial time computable envy-free allocation with subsidy payments (A,p) such that

1) Each agent gets at most one dollar of subsidy

2) Allocation A is balanced

3) Allocation A is EF1

Above implies subsidy of n-1 suffices

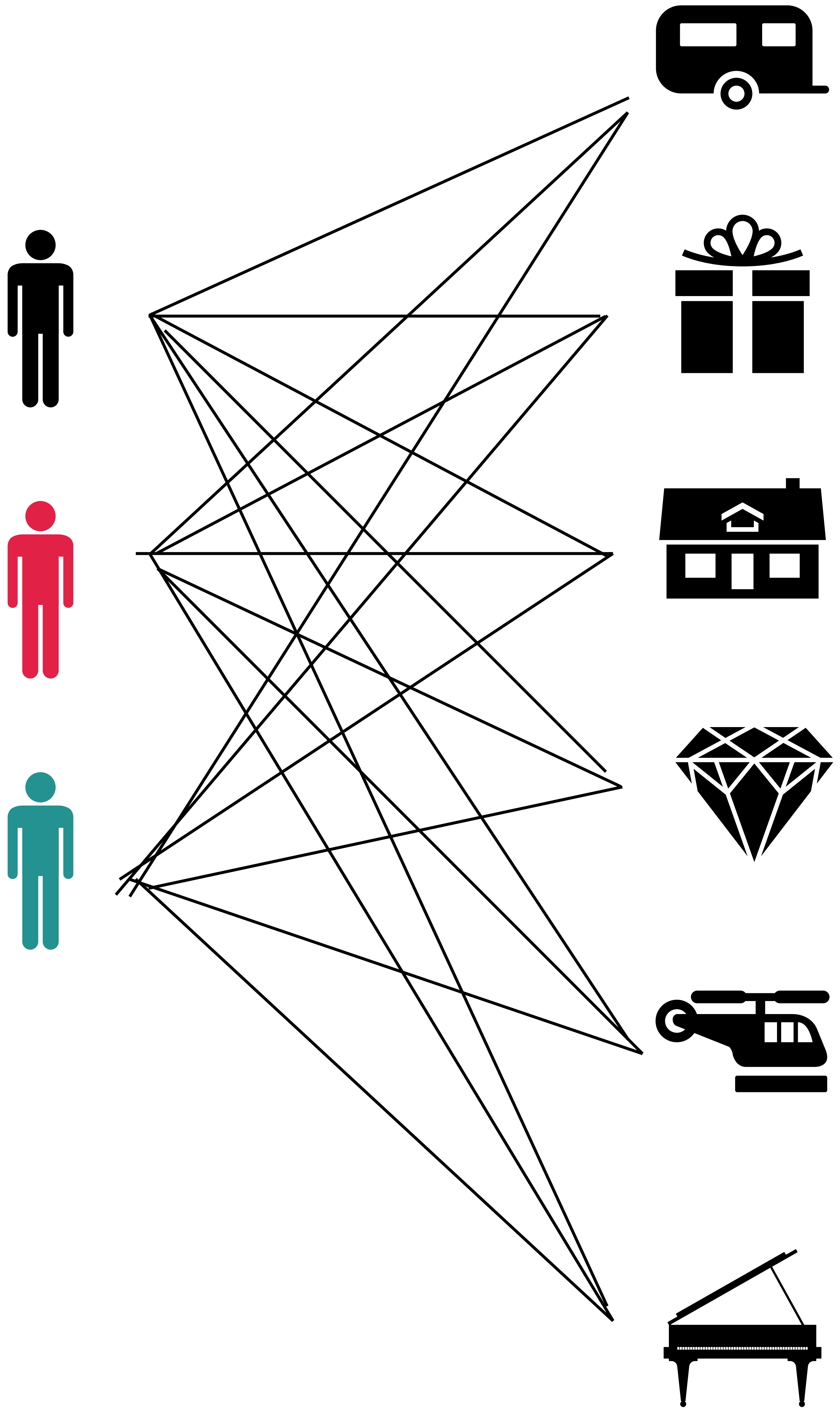
Iterated Max Weight Matching Algorithm

Weighted Complete
Bipartite Graph

$$G = K_{n,m}$$

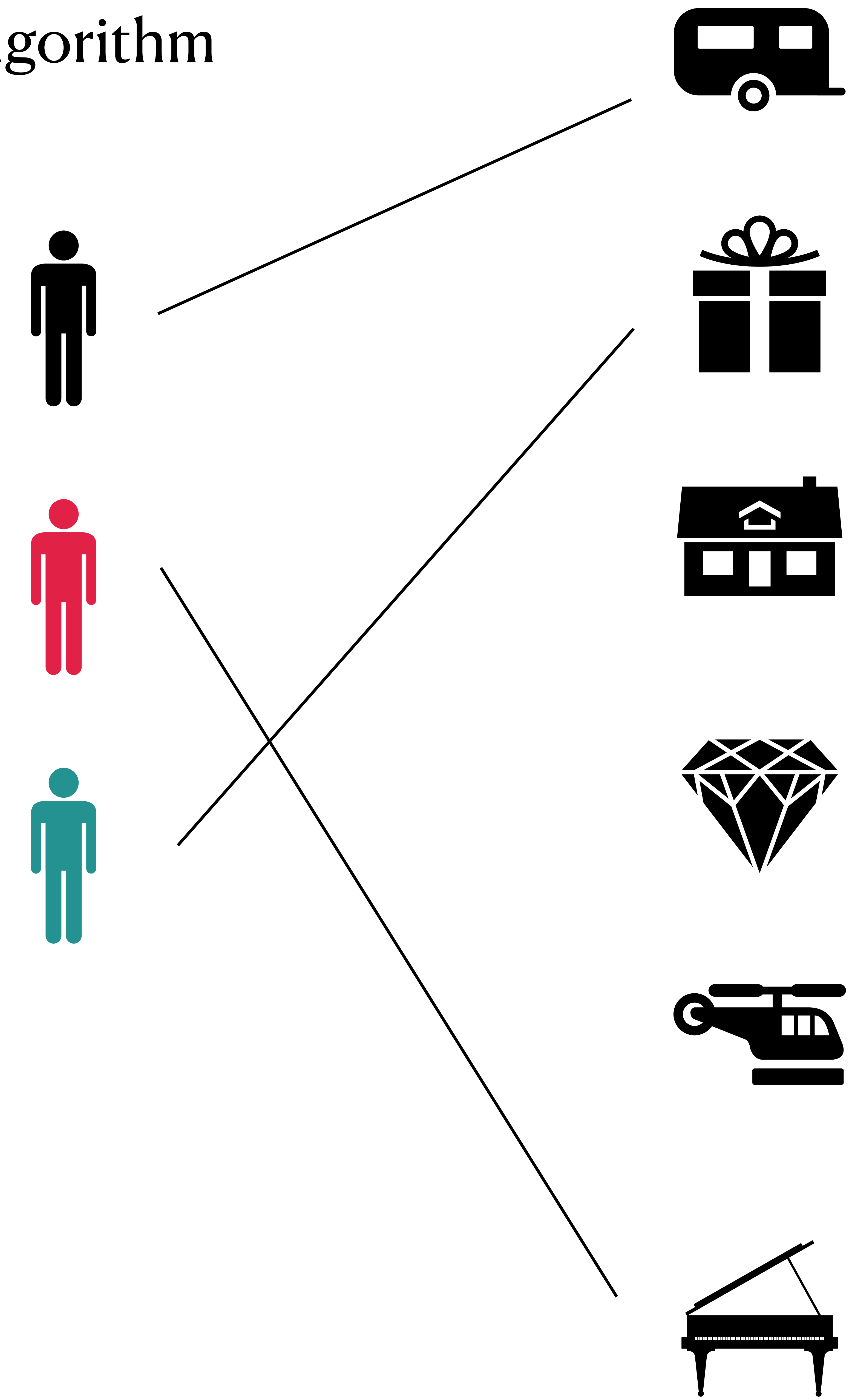
Edge Weights

$$w_{ij} = u_i(j) \quad \forall (i,j) \in E(K_{n,m})$$

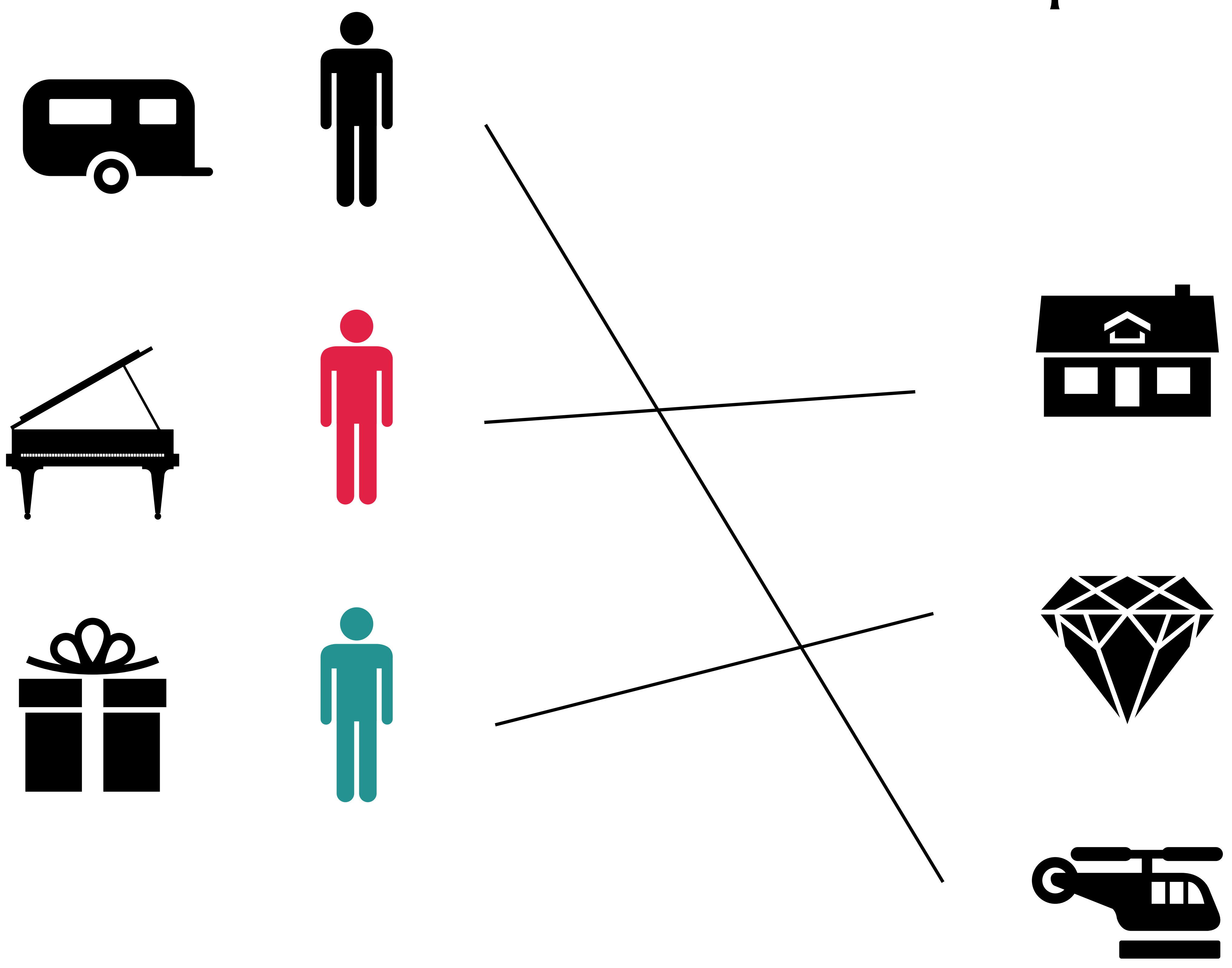


Iterated Max Weight Matching Algorithm

Compute Max Weight Matching



Repeated Max Weight Matching Algorithm



Repeated Max Weight Matching Algorithm

Final Allocation

Repeated Max Weight Matching Algorithm

Final Allocation

Although the algorithm itself is simple the analysis of the algorithm is quite involved!

What About Beyond Additive Valuations!

Theorem (Brustle, Dippel, Narayan, Suzuki, Vetta 20'):
For general valuations, there exist an envy-freeable allocation with total subsidy at most $2n^2$.
Given a valuation oracle, this allocation can be computed in polynomial time

Closing the Gap $2n^2$ and n-1

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- Subsidy of n-1 suffice for binary submodular functions.
Hiromichi Goko, Ayumi Igarashi, Yasushi Kawase, Kazuhisa Makino, Hanna Sumita, Akihisa Tamura, Yu Yokoi, and M. Yokoo. “Fair and truthful mechanism with limited subsidy”, 2021.

- Subsidy of n-1 suffice for dichotomous valuations.
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Is there an envy-free allocation with subsidy at most n-1 for any valuation function?

Thank You