Fair Lotteries for Collective Choice*

Haris Aziz[1], Xinhang Lu[1], Mashbat Suzuki[1], Jeremy Vollen[2], and Toby Walsh[1]

[1]UNSW Sydney, {haris.aziz, xinhang.lu, mashbat.suzuki, t.walsh}@unsw.edu.au
[2]Northwestern University, vollen@northwestern.edu

Abstract

We study a hierarchy of *collective choice* settings, including participatory budgeting (PB) and committee voting under both binary and general additive utilities. In these settings, fairness guarantees are typically limited to groups of voters with sufficiently cohesive preferences. In pursuit of outcomes with broader fairness guarantees, we study lotteries over discrete outcomes in our collective choice settings. Since we are the first to study randomization for PB, we must first address the question of implementation. As the projects have heterogeneous costs, the amount spent may not be equal ex ante and ex post. To address this, we develop a technique to bound the amount by which the ex-post spend differs from the ex-ante spend—the property is termed *budget balanced up to one project (BB1)*. We then introduce a hierarchy of *ex-ante* fairness properties based on the idea of fair share, including Individual Fair Share (IFS), Unanimous Fair Share (UFS) and their stronger variants, as well as Group Fair Share (GFS)—all of which are guaranteed to exist and retain a natural interpretation in each of the settings we study. Initiating the best-of-both-worlds perspective on fairness in collective choice, we pursue ex-ante and ex-post fairness simultaneously, drawing upon the extensive body of work on ex-post fairness concepts based on *justified representation*. In each of the five collective choice settings we study, we chart the compatibility landscape between our defined ex-ante properties and the existing ex-post fairness properties by giving explicit algorithms and complementary impossibility results. For instance, in the PB setting with binary utilities (over projects), we give a randomized algorithm which simultaneously satisfies ex-ante Strong UFS, ex-post full justified representation (FJR) and ex-post BB1. When voters have cost utilities, we can additionally satisfy a stronger property which captures both Strong UFS and GFS. Since these algorithms may require exponential time, we additionally provide polynomial-time algorithms with slightly weaker ex-post guarantees.

1 Introduction

A budget-constrained collective must select a subset of costly alternatives, accounting for each member's (or voter's) preferences over the alternatives. From a computer science perspective, this problem is aptly described as a multi-agent variant of the knapsack problem. Perhaps more concretely, the problem mirrors that which is solved annually by *participatory budgeting (PB)* processes

*This paper unifies and expands earlier versions that appeared in *Proceedings of the 19th Conference on Web and Internet Economics (WINE)* [Aziz et al., 2023b] and *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)* [Aziz et al., 2024b]. In particular, this version contains additional results in Section 5 "BoBW Fairness under Cost Utilities" and Section 7 "BoBW Fairness in General PB." Most of the work was done while the fourth author was at UNSW Sydney.

around the world. PB is a form of direct democracy that facilitates members of a community, municipality, or town to collectively make public project funding decisions, and has seen widespread adoption [Aziz and Shah, 2021; Rey et al., 2025]. As a special case, the described model also captures *committee voting*, wherein a collective must select a fixed number of alternatives [Faliszewski et al., 2017; Lackner and Skowron, 2023]. Both PB and committee voting pose interesting axiomatic and algorithmic research challenges. A major effort underway in computational social choice is to design meaningful axioms that capture elusive properties such as fairness and representation in these contexts, and to design computationally efficient algorithms that satisfy such axioms. In this work, we will investigate both of these models under various preference domains. We refer to the union of these decision-making settings simply as *collective choice*.

To design algorithms that reward engagement and provide some satisfaction to all voters, we would ideally like to guarantee that each voter receive some non-trivial representation from the selected outcome. While this goal is immediately thwarted by impossibilities inherited from the classical voting setting, it is straightforward to observe that much more can be achieved with respect to representation when fractional solutions are permitted, i.e., allowing alternatives to be fractionally selected. By randomizing over integral outcomes, we can produce a *lottery* (i.e., probability distribution) which corresponds to a given fractional outcome in an *ex-ante* sense. The use of randomization has been employed to achieve strong ex-ante fairness properties in various contexts, such as apportionment [Grimmet, 2004], resource allocation [Bogomolnaia and Moulin, 2001], voting [Bogomolnaia et al., 2005], and committee voting [Cheng et al., 2020].

In this work, we employ randomization in pursuit of more representatively fair outcomes across a broad hierarchy of collective choice settings. There are three main challenges we must tackle to achieve this goal. First, because we initiate the study of randomization for PB, we must answer the question of implementation: *Given a marginal probability for each project, how can we compute a probability distribution (or lottery) over discrete outcomes that realizes these probabilities and does not over or under spend too much?* While this question has already been answered in the PB special case of committee voting using various rounding techniques [e.g., Aziz et al., 2019; Gandhi et al., 2006; Grimmet, 2004], the presence of heterogeneous costs in the PB setting gives rise to significant obstacles. In particular, unlike committee voting, we cannot guarantee that the total amount of budget spent is equal ex ante and *ex post*.

Our second challenge is to introduce meaningful ex-ante fairness properties in each of the settings we study. We aim to design our properties so that (i) an outcome satisfying our properties is guaranteed to exist in even the most general setting we consider, and (ii) the properties implied by each of the special cases we consider remain natural and well-motivated. Lastly, we turn to the task of designing fair randomized algorithms for collective choice. In addition to the ex-ante properties we will define, there is a wealth of research on ex-post fairness properties in the settings we study, primarily focusing on the hierarchy of *justified representation* axioms, including strong properties like *extended justified representation (EJR)* [Aziz et al., 2017] and *full justified representation (FJR)* [Peters et al., 2021]. These properties give proportional representation guarantees to groups of voters whose preferences are in sufficient agreement. We take the view that it is not sufficient to achieve ex-ante fairness since this still allows the realization of blatantly unfair outcomes, and thus pursue ex-ante and ex-post fairness simultaneously. This approach is known as the *best-of-both-worlds fairness* perspective, and has been employed in adjacent contexts, such as resource allocation [e.g., Aziz et al., 2024a; Bu et al., 2024]. We are the first to apply this approach to collective choice, giving algorithms which guarantee best-of-both-worlds fairness and/or incompatibility results for each setting considered.

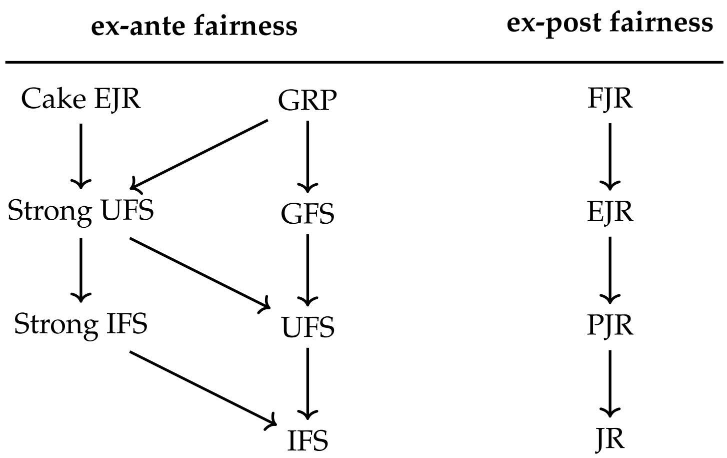

Figure 1: Visualization of ex-ante and ex-post fairness hierarchies studied for ABC voting. Besides the ex-ante notions defined in Section 4.1, cake EJR and ex-ante GRP were introduced and studied by Bei et al. [2025] and Suzuki and Vollen [2024], respectively. An arrow from (A) to (B) denotes that (A) implies (B). Any pair of ex-ante fairness notions not given explicit logical relation are logically independent.

1.1 Our Contributions

Our first contribution is to broaden the best-of-both-worlds fairness paradigm, which has so far been limited to resource allocation, and explore it in the context of social choice problems, specifically committee voting and participatory budgeting.

In Section 3, we tackle the question of implementation in PB. We first show that, unlike the committee voting setting, fractional outcomes cannot always be implemented by a lottery over integral outcomes using the same amount of budget. Given this, we define a well-motivated axiom for lotteries — *budget balanced up to one (BB1)* — which enforces a natural bound on the amount by which the total ex-post cost can differ from the cost of the fractional outcome the lottery implements. We then demonstrate an approach which gives an implementation satisfying our axiom for any fractional outcome, and complement this result by showing that a lottery satisfying a natural "up to any" strengthening of our axiom may not always exist.

Whereas ex-post fairness properties formalizing proportional representation have been examined at great length for both committee voting [Lackner and Skowron, 2023] and PB [Rey et al., 2025], the literature on ex-ante fairness is less developed. In Section 4, we formalize a hierarchy of ex-ante proportional representation properties that are careful extensions of similar concepts proposed in the restricted setting of single-winner voting. More specifically, we introduce the following concepts in increasing order of strength as well as their stronger variants: *individual fair share (IFS)*, *unanimous fair share (UFS)*, and *group fair share (GFS)*. We demonstrate in Figure 1 ex-ante and ex-post fairness hierarchies for approval-based committee (ABC) voting and establish logical relations between them. The "fair share" hierarchy of fairness axioms begins with the basic notion that each voter should receive at least a $1/n$ fraction of their optimal utility. On the other hand, the "strong fair share" hierarchy starts with the stronger guarantee that each voter should be able to control their proportion of the budget. At the other end of the spectrum, GFS gives a desirable level of ex-ante representation to *every* coalition of voters.

In line with the goals of the best-of-both-worlds fairness paradigm, our central research question is to understand which combinations of ex-ante and ex-post fairness properties can be achieved

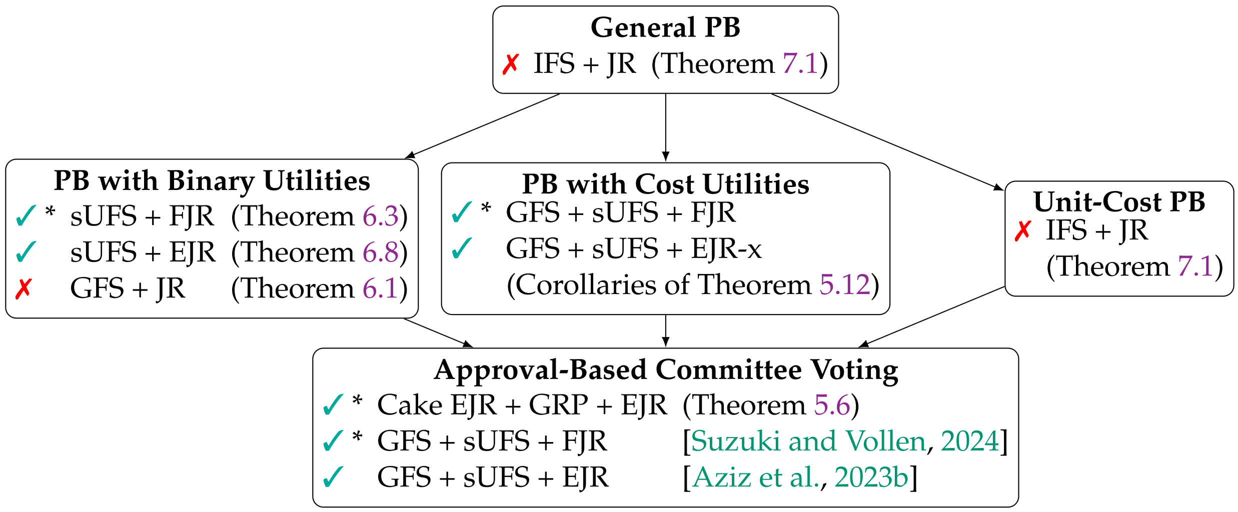

Figure 2: Summary of best-of-both-worlds fairness results in PB and special cases. Arrows point from generalizations to special cases. sUFS is used to abbreviate Strong UFS. Compatibility results are represented by ✓ and impossibility results by ✗. The asterisk (*) denotes exponential-time results.

simultaneously in the context of approval-based committee voting (Section 5.1) and various settings of participatory budgeting (Sections 5.2, 6 and 7). Our best-of-both-worlds fairness results for a variety of collective choice settings are summarized in Figure 2. More detailed descriptions of the results can be found below.

Section 5.1 is concerned with the special case of approval-based committee (ABC) voting. We show compatibility between ex-ante Strong UFS and ex-post FJR and between ex-ante Strong UFS, ex-ante GFS, and ex-post EJR in the work by Aziz et al. [2023b]. Suzuki and Vollen [2024] have subsequently improved on these results using a network flow approach. They define a new ex-ante fairness property called *group resource proportionality (GRP)* and show that ex-ante GRP is compatible with both EJR and FJR. Critically, since GRP is stronger than both Strong UFS and GFS, these results demonstrate compatibility between ex-ante GFS, ex-ante Strong UFS, and ex-post FJR. We then show in this paper that GRP and cake EJR[1] are compatible and, moreover, can be achieved alongside ex-post EJR. In Section 5.2, we extend the approach of Suzuki and Vollen [2024] and generalize those aforementioned results to the PB setting with cost utilities.

In Section 6, we investigate the special case of PB with binary utilities. We show that our strongest ex-ante fairness notion (i.e., GFS) cannot be guaranteed in tandem with any of our ex-post fairness notions (Section 6.1), notable since this is not the case in committee voting. We then give a strong, positive result, i.e., the compatibility between ex-ante Strong UFS and ex-post FJR, using an exponential-time algorithm (Section 6.2) and a slightly weaker positive result, i.e., the compatibility between Strong UFS and ex-post EJR, using a polynomial-time algorithm (Section 6.3).

Lastly, in Section 7, we show that in the general model with cardinal utilities, ex-ante and ex-post fairness are not compatible, even for the weakest pair of axioms and even in the restricted case where projects are of unit cost. Nevertheless, if we relax ex-ante fairness to guarantee strictly positive expected utility to each voter — a property known as *positive share*, ex-post FJR can be achieved simultaneously.

[1] Cake EJR [Bei et al., 2025] is a fractional analogue of EJR applied to fractional committees; see also Definition 4.4.

1.2 Related Work

There is a fast growing body of work investigating proportionally representative outcomes in indivisible PB [e.g., Aziz et al., 2018; Brill et al., 2023; Los et al., 2022; Munagala et al., 2022; Peters et al., 2021; Rey et al., 2025]. Since PB generalizes committee voting [Lackner and Skowron, 2023], work on proportionality in PB often extends axioms and algorithms from the literature on proportional representation in committee voting [e.g., Aziz et al., 2017; Brill and Peters, 2023; Elkind et al., 2024; Peters and Skowron, 2020; Sánchez-Fernández et al., 2017]. It is to this literature that we are adding the tool of randomization.

Aziz [2019] proposed research directions regarding probabilistic decision making with desirable ex-ante and ex-post stability or fairness properties. Aziz et al. [2024a] were the first to examine the compatibility of achieving ex-ante envy-freeness and ex-post approximate envy-freeness in the context of resource allocation and coin the term "best-of-both-worlds fairness." Since then, there have been a couple of papers on best-of-both–worlds fairness in resource allocation, including settings where agents have different entitlements [Aziz et al., 2023a; Hoefer et al., 2024] or have valuations beyond additive [Feldman et al., 2024; Kavitha et al., 2025], where resources consist of both divisible and indivisible goods [Bu et al., 2024], and where share-based fairness notions are studied [Akrami et al., 2023, 2024; Babaioff et al., 2022].

In the paper by Aziz et al. [2023b], we applied the best-of-both-worlds fairness perspective to the social choice setting of approval-based committee voting. To this end, we formalized several natural axioms for ex-ante fairness based on "fair shares" that are careful extensions of similar concepts proposed in the restricted setting of single-winner voting [Bogomolnaia et al., 2005]. Suzuki and Vollen [2024] later proposed a new, stronger ex-ante fairness notion called *group resource proportionality (GRP)* and provided an improved best-of-both-worlds fairness result by showing the compatibility between ex-ante GRP and ex-post FJR (a demanding proportional representation axiom for deterministic committees). Kehne et al. [2025] introduced several ex-ante candidate fairness concepts, including *neutrality*, *monotonicity* and *continuity*, and presented a randomized voting rule that satisfies all three aforementioned candidate-fair notions while maintaining *EJR+* (a strengthening of EJR) ex post. In the committee voting setting where agents have ordinal preferences, Peters [2025] defined an ex-ante version of *proportionality for solid coalitions (PSC)* and showed it can be satisfied together with (ex-post) PSC.

In a follow-up paper by Aziz et al. [2024b], we studied lotteries in multiple PB settings, all of which generalize ABC voting. Here, we extended the hierarchy of (strong) fair shares properties from the ABC voting to our most general PB setting. Papasotiropoulos et al. [2025] defined a fractional analogue of EJR in the setting of general PB called *Fractional EJR*,[2] and showed Fractional Equal Shares (FrES), an adaption of the Method of Equal Shares (MES) to fractional PB, satisfies Fractional EJR. Papasotiropoulos et al. mainly built upon the idea behind FrES and obtained a variant of MES for indivisible PB where voters may occasionally overspend.

Lottery implementation techniques have been studied in other social choice settings. For example, the dependent randomized rounding technique of Gandhi et al. [2006] has been employed to compute randomized outcomes which implement desirable fractional outcomes in various settings such as resource allocation [Akbarpour and Nikzad, 2020] and committee selection [Cheng et al., 2020; Munagala et al., 2022]. In the context of apportionment, Aziz et al. [2019] and Gölz et al. [2025] have created new rounding techniques to facilitate randomization when distributing legislative seats.

There have also been several papers which study divisible PB, wherein projects can be funded

[2] Papasotiropoulos et al. [2025] remarked that when applied to approval ballots with cost utilities, Fraction EJR is equivalent to cake EJR.

fractionally. Fain et al. [2016] showed that the Nash solution in this setting is in the *core*, a property which captures proportional representation. In this paper and most work in this area, projects do not have fixed costs, and thus any distribution of budget amongst projects is feasible, which contrasts significantly with our setting. Goel et al. [2019] incorporated project costs in the divisible PB setting and study strategic concerns. However, their outcomes still allow for fractional project funding. In this paper, we investigate the question of how such an outcome can be converted into a lottery over outcomes in which funding decisions are binary.

2 Preliminaries

For any positive integer $t \in \mathbb{N}$, let $[t] := \{1, 2, \ldots, t\}$. A participatory budgeting (PB) *instance* is represented as a tuple $I = \langle N, C, \mathrm{cost}, B, (u_i)_{i\in[n]} \rangle$, where:

- $N := [n]$ and $C := [m]$ are the set of *voters* (or *agents*) and *projects* (or *candidates*), respectively.
- $\mathrm{cost}\colon C \to \mathbb{R}_{\geq 0}$ is the *cost function*, associating each project $c \in C$ with a cost that needs to be paid if c is selected. For any subset of projects $T \subseteq C$, denote by $\mathrm{cost}(T) := \sum_{c\in T} \mathrm{cost}(c)$ the total cost of T.

  We say projects have *unit costs* if $\mathrm{cost}(c) = 1$ for all $c \in C$, and refer to the setting as *unit-cost PB*. This setting is also referred to as *committee voting* in the literature.
- $B \in \mathbb{R}_{\geq 0}$ is the *budget limit*. We assume without loss of generality that $\mathrm{cost}(c) \leq B$ for each $c \in C$ and $\mathrm{cost}(C) \geq B$. When concerning committee voting, following the convention of the literature, we will use positive integer k (instead of B) to denote the committee size.
- For each voter $i \in N$, *utility function* $u_i\colon C \to \mathbb{R}_{\geq 0}$ expresses how voter i values each project. We call the set of projects for which voter i has non-zero utility their *approval set* and denote it as A_i. This most general formulation is referred to as *general PB*.

  We say voters have *binary utilities* if $u_i(c) \in \{0, 1\}$ for all $i \in N, c \in C$, and refer to the setting as *PB with binary utilities*.

  Another interesting special case is to assume that voters have *cost utilities*, that is, $u_i(c) \in \{0, \mathrm{cost}(c)\}$ for each $i \in N$ and $c \in C$. We refer to this setting as *PB with cost utilities*.

When candidates have unit costs (and budget B is an integer) and voters have binary utilities over the candidates, this setting is often referred to as *approval-based committee (ABC) voting* (or *approval-based multi-winner voting*), which has been attracting significant attention and interest in recent years [Lackner and Skowron, 2023]. Furthermore, most ex-post proportional representation notions focused in this paper were first proposed in ABC voting, and later have been generalized to the setting of general PB (see Section 2.1).

Integral Outcomes An *integral outcome* (or simply *outcome*) is a set of projects $W \subseteq C$, and it is said to be *feasible* if $\mathrm{cost}(W) \leq B$. We assume *additive* utilities, meaning that given a subset of projects $T \subseteq C$, $u_i(T) := \sum_{c\in T} u_i(c)$.

Fractional Outcomes A *fractional outcome* is an m-dimensional vector $\vec{p} \in [0, 1]^m$, where the component $p_c \in [0, 1]$ represents the fraction of project c funded. Given an integral outcome W, for notational convenience, let $\vec{1}_W \in \{0, 1\}^m$ be the binary vector whose j-th component is 1 if and only if $j \in W$. Let $\mathrm{cost}(\vec{p}) := \sum_{c\in C} p_c \cdot \mathrm{cost}(c)$ denote the total cost of $\vec{p}$. A fractional outcome $\vec{p}$ is said to be *feasible* if $\text{cost}(\vec{p}) = B$. Given budget B', let $\mathcal{X}(B')$ denote the space of all feasible fractional outcomes under budget B'. For simplicity, we will use $\mathcal{X}$ to denote the space of all feasible fractional outcomes given budget B. Given a fractional outcome $\vec{p}$, voter i's utility is denoted by $u_i(\vec{p}) := \sum_{c \in C} p_c \cdot u_i(c)$.

Lotteries and Implementation A *lottery* (or *randomized outcome*) is a probability distribution over integral outcomes. Formally, a lottery is specified by a set of $s \in \mathbb{N}$ tuples $\{(\lambda_j, W_j)\}_{j \in [s]}$, where $\lambda_j \in [0, 1]$, $\sum_j \lambda_j = 1$, and for every $j \in [s]$, the integral outcome $W_j \subseteq C$ is selected with probability λ_j. A lottery $\{(\lambda_j, W_j)\}_{j \in [s]}$ is called an *implementation* of (or, interchangeably, *implements*) a fractional outcome $\vec{p}$ if $\vec{p} = \sum_{j \in [s]} \lambda_j \cdot \vec{1}_{W_j}$. In this paper, we only consider lotteries which implement feasible fractional outcomes. We say a lottery satisfies a property ex ante (resp., ex post) if the fractional outcome it implements (resp., every integral outcome in its support) satisfies the property.

2.1 Fairness for Integral Outcomes

Fair representation axioms for integral outcomes are well-studied in the hierarchy of collective choice settings considered in this paper. A hierarchy of desiderata that has received significant attention is based on *justified representation (JR)*, which was first introduced by Aziz et al. [2017] in the context of approval-based committee voting. The idea behind JR and its strengthenings is that a large-enough group of voters with similar preferences (i.e., a cohesive group) deserves to be satisfied with a certain number of representatives in the selected committee that is proportional to the size of the group.

In what follows, we will start by defining JR and its strengthenings such as *proportional justified representation (PJR)* [Sánchez-Fernández et al., 2017] and *extended justified representation (EJR)* [Aziz et al., 2017] for ABC voting, followed by providing their adaptations to PB with binary utilities and to general PB (where we will also define *full justified representation (FJR)* [Peters et al., 2021]).

Definition 2.1 (JR, PJR & EJR in ABC voting)**.** Given an instance of ABC voting, for any positive integer ℓ, a group of voters $N' \subseteq N$ is said to be *ℓ-cohesive* if $|N'| \geq \ell \cdot \frac{n}{k}$ and $|\bigcap_{i \in N'} A_i| \geq \ell$.[3]

An integral committee W is said to satisfy

- *JR* if for every 1-cohesive group of voters $N' \subseteq N$, it holds that $A_i \cap W \neq \emptyset$ for some $i \in N'$;
- *PJR* if for every positive integer ℓ and every ℓ-cohesive group of voters $N' \subseteq N$, it holds that $|(\bigcup_{i \in N'} A_i) \cap W| \geq \ell$;
- *EJR* if for every positive integer ℓ and every ℓ-cohesive group of voters $N' \subseteq N$, it holds that $|A_i \cap W| \geq \ell$ for some $i \in N'$.

It follows directly from the definitions that EJR implies PJR, which in turn implies JR. A committee providing EJR (and therefore PJR and JR) always exists and can be computed in polynomial time [Aziz et al., 2017; Peters and Skowron, 2020]. These concepts have been examined in participatory budgeting with arbitrary costs and arbitrary additive utilities (i.e., general PB). We first provide relevant adaptions of the above proportional representation notions for PB with binary utilities, that is, the voters have binary utilities while the projects have arbitrary costs.

[3] Recall that we use positive integer k (instead of B) to denote committee size.

Definition 2.2 (JR & EJR for PB with binary utilities)**.** Given an instance of PB with binary utilities, a group of voters $S \subseteq N$ is said to be *T-cohesive* for $T \subseteq C$ if $|S| \cdot \frac{B}{n} \geq \text{cost}(T)$ and $T \subseteq \bigcap_{i \in S} A_i$.

An outcome W is said to satisfy

- *JR* if for each $j \in C$ and every $\{j\}$-cohesive group of voters $S \subseteq N$, it holds that $u_i(W) = |A_i \cap W| \geq 1$ for some $i \in S$;
- *EJR* if for each $T \subseteq C$ and every T-cohesive group of voters $S \subseteq N$, it holds that $u_i(W) = |A_i \cap W| \geq |T|$ for some $i \in S$.

An outcome providing EJR (and therefore JR) always exists and can be computed in polynomial time using the *Method of Equal Shares (MES)* [Peters et al., 2021]. Finally, we define proportional representation notions for general PB.

Definition 2.3 (JR & EJR in PB [Peters et al., 2021])**.** Given an instance of general PB, a group of voters $S \subseteq N$ is said to be *(α, T)-cohesive*, for a function $\alpha \colon C \to \mathbb{R}_{\geq 0}$ and a set of projects $T \subseteq C$, if $|S| \cdot \frac{B}{n} \geq \text{cost}(T)$ and $u_i(j) \geq \alpha(j)$ for all $i \in S$ and $j \in T$.

An integral outcome W is said to satisfy

- *JR* if for each $\alpha \colon C \to \mathbb{R}_{\geq 0}$, $j \in C$, and each $(\alpha, \{j\})$-cohesive group of voters $S \subseteq N$, there exists a voter $i \in S$ such that $u_i(W) \geq \alpha(j)$;
- *EJR* if for each $\alpha \colon C \to \mathbb{R}_{\geq 0}$, $T \subseteq C$, and each (α, T)-cohesive group of voters $S \subseteq N$, there exists a voter $i \in S$ such that $u_i(W) \geq \sum_{j \in T} \alpha(j)$.

In the setting of general PB, MES no longer always output an EJR outcome. Nevertheless, for every instance of general PB, an EJR outcome does exist and can be computed using the (computationally intractable) *Greedy Cohesive Rule (GCR)* [Peters et al., 2021]. As a matter of fact, the outcome output by GCR even satisfies a stronger notion which is defined below.

Definition 2.4 (FJR in PB [Peters et al., 2021])**.** Given an instance of general PB, a group of voters $S \subseteq N$ is said to be *weakly (β, T)-cohesive* for $\beta \in \mathbb{R}$ and $T \subseteq C$, if $|S| \cdot \frac{B}{n} \geq \text{cost}(T)$ and $u_i(T) \geq \beta$ for every voter $i \in S$.

An outcome W is said to satisfy *full justified representation (FJR)* if for every weakly (β, T)-cohesive group of voters $S \subseteq N$, it holds that $u_i(W) \geq \beta$ for some $i \in S$.

This paper concerns the problem of designing (randomized) PB rules (or, interchangeably, an algorithm) that simultaneously achieve desirable properties both ex ante and ex post. We start in Section 3 by addressing how to implement a fractional outcome in the context of participatory budgeting, followed by formalizing ex-ante fairness concepts in Section 4. All remaining sections are then devoted to studying to what extent the best-of-both-worlds fairness can be achieved in the contexts of approval-based committee voting (Section 5.1), PB with cost utilities (Section 5.2), PB with binary utilities (Section 6), and general PB (Section 7).

3 Implementing Fractional Outcomes

Let us first restrict ourselves to the setting of unit-cost PB, where the budget B is an integer and each project has a cost of exactly 1. The fact that any fractional outcome can be implemented by a probability distribution over integral outcomes of the same size is implied by various works on randomized rounding schemes in combinatorial optimization. We explain this connection explicitly using the classical result of Gandhi et al. [2006] and frame it in our context. Theorem 2.3 of

Gandhi et al. [2006] states that there is a polynomial-time rounding scheme to *sample* an integral outcome from a lottery satisfying three properties. The first property ensures that the lottery is a valid implementation of the fractional outcome. The second property ensures that each integral outcome in the support of the implementation are of size B. We do not need the third property for our purposes.

While one can sample an outcome from a support in polynomial time, the support could have exponential size. In scenarios where we want an explicit construction of a lottery, such a rounding scheme does not run in polynomial time due to the exponential-size output. In unit-cost PB, in order to have polynomial-time computation, we can resort to randomized rounding schemes which output an explicit probability distribution over a support of polynomial size, for example, the ALLOCATIONFROMSHARES of Aziz et al. [2019].

Unlike the case of unit-cost PB, decomposing a fractional outcome into a distribution over integral outcomes introduces novel challenges in the presence of *heterogeneous* costs. Recall that implementing a fractional outcome $\vec{p}$ entails computing a probability distribution over integral outcomes, denoted as $\Delta = \{(\lambda_j, W_j)\}_{j\in[s]}$, that realizes marginal probability p_c for each project $c \in C$. As a result, any implementation of a feasible fractional outcome has the property that

$$\mathbb{E}_{W\sim\Delta}[\text{cost}(W)] = B.$$

It is now easy to see that unless each integral outcome in the support of the lottery has cost equal to B (not possible in general), there must exist an integral outcome in the support of the lottery that exceeds the budget.

The aforementioned issue raises the natural question of whether it is possible to implement a fractional outcome while bounding the ex-post budget violations. This is especially important in participatory budgeting since if the ex-post budget constraint is exceedingly violated, such an outcome is unlikely to be implemented in practice. To this end, we formalize an axiom which guarantees that an integral outcome is approximately within budget.

Definition 3.1 (BB1)**.** An integral outcome W is said to be *budget balanced up to one project (BB1)* if either

- $\text{cost}(W) \leq B$ and there exists some project $c \in C \setminus W$ such that $\text{cost}(W \cup \{c\}) \geq B$, or
- $\text{cost}(W) \geq B$ and there exists some project $c \in W$ such that $\text{cost}(W \setminus \{c\}) \leq B$.

We now show, perhaps surprisingly, that *any* feasible fractional outcome $\vec{p}$ can be implemented by a lottery, and each integral outcome in its support satisfies BB1.

Theorem 3.2. *For any feasible fractional outcome $\vec{p}$, there exists a random process running in polynomial time, that defines random variables $P_i \in \{0,1\}$ for all $i \in C$ such that the following properties hold:*

(P1) *$\mathbb{E}[P_i] = p_i$ for each $i \in C$;*

(P2) *Random integral committee $W = \{i \in C \mid P_i = 1\}$ satisfies BB1 with probability 1.*

Proof. This proof follows from applying the dependent rounding technique introduced by Gandhi et al. [2006]. Given a fractional outcome $\vec{p} = (p_1, \ldots, p_m)$ with $p_i \in [0,1]$, we probabilistically modify each p_i to a random variable $P_i \in \{0,1\}$ such that the random variables satisfy the properties **(P1)** and **(P2)**.

We now describe the algorithm. Let $\vec{q}^{\,0} = \vec{p}$. We iteratively and randomly modify $\vec{q}^{\,0}$ in rounds. Denote $\vec{q}^{\,t} = (q_1^t, q_2^t, \ldots, q_m^t)$ as the values at round t. In each round, we update the values of at most two indices while keeping the values of all other indices constant. Let $F^t = \{i \in C \mid q_i^t \in (0,1)\}$ be the set of indices that are fractional in round t. The update rule depends on the cardinality of F^t.

If $|F^t| \geq 2$, we arbitrarily select two indices $i, j \in F^t$ and run the following randomized update rule:

$$(q_i^{t+1}, q_j^{t+1}) = \begin{cases} \left(q_i^t + \alpha, q_j^t - \frac{\text{cost}(i)}{\text{cost}(j)} \cdot \alpha\right) & \text{w.p. } \frac{\beta}{\alpha+\beta} \\ \left(q_i^t - \beta, q_j^t + \frac{\text{cost}(i)}{\text{cost}(j)} \cdot \beta\right) & \text{w.p. } \frac{\alpha}{\alpha+\beta} \end{cases}$$

where

$$\alpha = \min\{\gamma > 0 \mid q_i^t + \gamma = 1 \text{ or } q_j^t - \frac{\text{cost}(i)}{\text{cost}(j)} \cdot \gamma = 0\},$$

and

$$\beta = \min\{\gamma > 0 \mid q_i^t - \gamma = 0 \text{ or } q_j^t + \frac{\text{cost}(i)}{\text{cost}(j)} \cdot \gamma = 1\}.$$

For all other indices $\ell \in C \setminus \{i, j\}$, we set $q_\ell^{t+1} = q_\ell^t$.

If $|F^t| = 1$, we select the fractional index $\ell \in F^t$ and set $q_\ell^{t+1} = 1$ with probability q_ℓ^t, and $q_\ell^{t+1} = 0$ with probability $1 - q_\ell^t$.

Finally, when no fractional indices exist, meaning $|F^t| = 0$ and hence $q_i^t \in \{0,1\}$ for each $i \in C$, we terminate the algorithm and set $P_i = q_i^t$ for all $i \in C$.

The next two observations immediately follow from the algorithm's description.

Observation 1) After each round, at least one index with a fractional value becomes integral i.e., $|F^{t+1}| < |F^t|$.

Observation 2) Once a fractional index becomes integral, its values do not change.

Note that by the first observation and since $|F^0| \leq |C|$, we see that the number of rounds is at most $|C|$. As a result, we see that the random process does indeed run in polynomial time.

In what follows, we show the two properties and begin with the first property.

*Proof of **(P1)**.* For each $i \in C$, let P_i^t be the random variable denoting the value of q_i^t. We show that in each round, $\mathbb{E}[P_i^{t+1}] = \mathbb{E}[P_i^t]$ for each $i \in C$. The assertion trivially holds true for indices that were not updated, thus we focus on indices which were updated.

Suppose $|F^t| \geq 2$, and let i, j be the indices which were chosen in the update rule. Then for all $\zeta \in [0,1]$,

$$\mathbb{E}[P_i^{t+1} \mid P_i^t = \zeta] = \frac{\beta(\zeta + \alpha)}{\alpha + \beta} + \frac{\alpha(\zeta - \beta)}{\alpha + \beta} = \zeta;$$

$$\mathbb{E}[P_j^{t+1} \mid P_j^t = \zeta] = \left(\zeta - \frac{\text{cost}(i)}{\text{cost}(j)}\alpha\right)\frac{\beta}{\alpha + \beta} + \left(\zeta + \frac{\text{cost}(i)}{\text{cost}(j)}\beta\right)\frac{\alpha}{\alpha + \beta} = \zeta.$$

Let Ω be the set of all possible values of q_i^t, then we see that

$$\mathbb{E}[P_i^{t+1}] = \sum_{\zeta \in \Omega} \mathbb{E}[P_i^{t+1} \mid P_i^t = \zeta] \cdot \Pr[P_i^t = \zeta] = \sum_{\zeta \in \Omega} \zeta \cdot \Pr[P_i^t = \zeta] = \mathbb{E}[P_i^t].$$

An analogous argument can be used to show $\mathbb{E}[P_j^{t+1}] = \mathbb{E}[P_j^t]$.

Consider now the case in which the update rule is applied when $|F^t| = 1$. Let ℓ be the fractional index which was updated. Then for all $\zeta \in [0,1]$,

$$\mathbb{E}[P_\ell^{t+1} \mid P_\ell^t = \zeta] = 1 \cdot \zeta + 0 \cdot (1 - \zeta) = \zeta.$$

Thus, we get that $\mathbb{E}[P_\ell^{t+1}] = \mathbb{E}[P_\ell^t]$ by a similar argument as seen in the previous paragraph.

It follows that in each round $\mathbb{E}[P_i^{t+1}] = \mathbb{E}[P_i^t]$ for all $i \in C$. Let t_f be the round in which the algorithm terminates, recursively applying the identity we see that $\mathbb{E}[P_i] = \mathbb{E}[P_i^{t_f}] = \mathbb{E}[P_i^0] = p_i$ for each $i \in C$. □

We now show the second property.

Proof of ***(P2)****.* We need to show that the random integral outcome $W = \{i \in C \mid P_i = 1\}$ satisfies BB1 with probability 1. We first show that whenever the update rule is applied with $|F^t| \geq 2$, it holds that $\sum_{i \in C} \mathrm{cost}(i) \cdot q_i^{t+1} = \sum_{i \in C} \mathrm{cost}(i) \cdot q_i^t$. Suppose indices i, j were selected to be updated in round t. Then, regardless of the realization of the randomized update rule, the following holds:

$$\begin{aligned}\mathrm{cost}(i) \cdot q_i^{t+1} + \mathrm{cost}(j) \cdot q_j^{t+1} &= \begin{cases} \mathrm{cost}(i) \cdot (q_i^t + \alpha) + \mathrm{cost}(j) \cdot \left(q_j^t - \frac{\mathrm{cost}(i)}{\mathrm{cost}(j)} \cdot \alpha\right) \\ \qquad\qquad \text{or} \\ \mathrm{cost}(i) \cdot (q_i^t - \beta) + \mathrm{cost}(j) \cdot \left(q_j^t + \frac{\mathrm{cost}(i)}{\mathrm{cost}(j)} \cdot \beta\right) \end{cases} \\ &= \mathrm{cost}(i) \cdot q_i^t + \mathrm{cost}(j) \cdot q_j^t.\end{aligned}$$

As values of all other indices $C \setminus \{i, j\}$ were unchanged, we see that $\sum_{i \in C} \mathrm{cost}(i) \cdot q_i^{t+1} = \sum_{i \in C} \mathrm{cost}(i) \cdot q_i^t$. Thus, noting that $\sum_{i \in C} \mathrm{cost}(i) \cdot q_i^0 = \sum_{i \in C} \mathrm{cost}(i) \cdot p_i = B$, we have $\sum_{i \in C} \mathrm{cost}(i) \cdot q_i^{t+1} = B$ whenever the update rule is applied with $|F^t| \geq 2$. Note that in all rounds, except possibly the last round, we have $|F^t| \geq 2$. Hence we see that $\sum_{i \in C} \mathrm{cost}(i) \cdot q_i^{t_f - 1} = B$, where t_f is the round when the algorithm terminates.

In round $t_f - 1$, either the update rule is applied with $|F^{t_f - 1}| \geq 2$ or $|F^{t_f - 1}| = 1$. If the update rule is applied with $|F^{t_f - 1}| \geq 2$, then $\sum_{i \in C} \mathrm{cost}(i) \cdot q_i^{t_f} = \sum_{i \in C} \mathrm{cost}(i) \cdot q_i^{t_f - 1} = B$ and hence $\sum_{i \in C} \mathrm{cost}(i) \cdot P_i = B$, meaning that the random committee $W = \{i \in C \mid P_i = 1\}$ is BB1.

Now suppose that $|F^{t_f - 1}| = 1$ and let $\ell \in F^{t_f - 1}$. We may write,

$$\begin{aligned} B = \sum_{i \in C} \mathrm{cost}(i) q_i^{t_f - 1} &= \sum_{i \in C \setminus \ell} \mathrm{cost}(i) q_i^{t_f - 1} + \mathrm{cost}(\ell) q_\ell^{t_f - 1} \\ &= \sum_{i \in C \setminus \ell} \mathrm{cost}(i) q_i^{t_f} + \mathrm{cost}(\ell) q_\ell^{t_f - 1} \\ &= \sum_{i \in C \setminus \ell} \mathrm{cost}(i) P_i + \mathrm{cost}(\ell) q_\ell^{t_f - 1}. \end{aligned}$$

Here we used the fact that the update rule only changed the value of index ℓ in round $t_f - 1$, and thus $P_i = q_i^{t_f} = q_i^{t_f - 1}$ for all $i \in C \setminus \ell$. Recall $W = \{i \in C \mid P_i = 1\}$. If $q_\ell^{t_f} = 1$, then W satisfies BB1 since $\mathrm{cost}(W) \geq B$ and removing ℓ makes it under budget. Similarly, if $q_\ell^{t_f} = 0$, then $\mathrm{cost}(W \cup \{\ell\}) \geq B \geq \mathrm{cost}(W)$. Thus, we have that W satisfies BB1 with probability one. □

We have now established Theorem 3.2. □

Note that there is a lottery associated with the random process described in Theorem 3.2, but we only return an integral outcome sampled from this underlying lottery as it may be exponential in size. By **(P1)**, the underlying lottery implements $\vec{p}$, and by **(P2)**, it satisfies ex-post BB1. We remark that when concerning general PB, our algorithms do not explicitly output the desired lotteries (which in principle can be exponential in size), but instead sample integral outcomes from them.

It is tempting to further strengthen the ex-post budget feasibility axiom BB1. A natural strengthening is the following:

Definition 3.3 (BFx)**.** An integral outcome W is said to be *budget feasible up to any project (BFx)* if for all $c \in W$, $\text{cost}(W \setminus \{c\}) \leq B$.

It is worth noting that BFx is weaker than the natural "up to any" strengthening of BB1. In particular, BFx only bounds the amount an outcome can exceed the budget, and places no restriction on outcomes which are under budget. We, however, show that not all fractional outcomes may be implemented by ex-post BFx lotteries.

Proposition 3.4. *There exists some fractional outcome $\vec{p}$ that cannot be implemented by a lottery satisfying ex-post BFx.*

Proof. Consider an instance with budget B and three projects $C = \{a, b, c\}$ such that $\text{cost}(a) = \varepsilon$ and $\text{cost}(b) = \text{cost}(c) = \frac{B}{2} + \varepsilon$. Consider the fractional outcome $\vec{p} = \left(1, \frac{B-\varepsilon}{B+2\varepsilon}, \frac{B-\varepsilon}{B+2\varepsilon}\right)$. We now show that $\vec{p}$ cannot be implemented by ex-post BFx integral outcomes. First, since $p_1 = 1$, project a needs to be included in every integral outcome. Next, the integral outcome $C = \{a, b, c\}$ is not BFx because $\text{cost}(C \setminus \{a\}) = B + 2\varepsilon > B$. It means that the lottery can positive support only on integral outcomes $\{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}$. However, such a lottery cannot implement $\vec{p}$ as $\text{cost}(\vec{p}) = B$ and every integral outcome in the support of the lottery has cost strictly less than B. $\square$

Handling Hard Constraints We note that, in addition to being well-suited to scenarios in which budget constraints have some flexibility, the implementation techniques introduced in this section are also relevant to settings with hard ex-post budget constraints. To see this, consider a problem wherein every ex-post outcome is restricted to having a cost of at most B. If we now apply Theorem 3.2 to any fractional outcome that spends $B' = B - \max_{g \in C} \text{cost}(g)$, the resulting implementation has the property that every integral outcome in its support has cost at most B.

4 Ex-ante Fairness Concepts

In this section, we introduce fairness properties for fractional outcomes. Whereas the literature on fairness concepts for integral outcomes is very well-developed (see Section 2.1), fairness properties for fractional outcomes are largely unexplored except for the very special case of single-winner voting [Aziz et al., 2020; Bogomolnaia et al., 2005; Duddy, 2015]. Exceptions are the two papers by Bei et al. [2025] and Lu et al. [2024].[4] Bei et al. [2025] introduced a divisible analogue of the ABC voting called *cake sharing*, in which "cake" is referred to as a divisible resource over which agents have piecewise uniform (i.e., binary) utilities and a subset of the cake is chosen subject to a length constraint — this model is more general than fractional ABC voting. Lu et al. [2024] further studied a setting that simultaneously generalizes both ABC voting and cake sharing. The fairness concepts studied in those two papers are directly inspired by the hierarchy of JR axioms. We will provide connections between their notions and ours later.

As we will see shortly, it is non-trivial to generalize the properties proposed in single-winner voting to more general settings. We thus first introduce how we generalize those properties for

[4] Also, as we have mentioned in Section 1 "Introduction", after a conference version of this work published, Suzuki and Vollen [2024] proposed the ex-ante fairness property of group resource proportionality (GRP) for ABC voting.

ABC voting in Section 4.1, and next for general PB in Section 4.2. Each of the axioms given in this section provides lower bounds on utilities derived from fractional outcomes. These utilities can also be interpreted as expected utilities from implementations of these fractional outcomes. In particular, if $\Delta = \{(\lambda_j, W_j)\}_{j \in [s]}$ is an implementation of a fractional outcome $\vec{p}$, then $\mathbb{E}_{W \sim \Delta}[u_i(W)] = u_i(\vec{p})$.

4.1 Ex-ante Fairness for Approval-Based Committee Voting

In this subsection, we introduce a hierarchy of fairness notions for fractional committees in the setting of ABC voting (see Figure 1), inspired by the *fair share* axioms first introduced by Bogomolnaia et al. [2005] in the single-winner context. Recall that in ABC voting, each project is of unit cost and voters have binary utilities. Moreover, by adopting the notational convention in the committee voting literature, let positive integer k (instead of B) denote the committee size.

The weakest in the hierarchy of axioms is *individual fair share (IFS)*, the idea behind which is that "each one of the n voters 'owns' a $\frac{1}{n}$-th share of decision power, so she can ensure an outcome she likes with probability at least $1/n$" [Aziz et al., 2020, pp. 18:2]. This idea suggests at least two distinct interpretations of the utility lower bound guaranteed by IFS in ABC voting:

(a) Fair share: each voter is given $\frac{1}{n}$ probability to choose their favourite *integral* committee, or

(b) Strong fair share: each voter can select $\frac{1}{n}$ of the *fractional* committee.

In probabilistic voting, as we will see shortly, both interpretations coincide. Critically, this is not the case in committee voting. Instead, these interpretations diverge and lead to two alternative hierarchies of fair share axioms for ABC voting, which we term *fair share* and *strong fair share*, respectively.

We begin by defining both generalizations of IFS. Both impose a natural lower bound on individual utilities stronger than that of *positive share*, which only requires that $u_i(\vec{p}) > 0$. In the single-winner setting, IFS requires that the probability that the (single) alternative selected is approved by any individual voter is no less than $1/n$. It is thus tempting to require $u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \frac{k}{n}$, which turns out to be too strong in our setting as a fractional committee satisfying it may not exist.[5] Intuitively speaking, this is because our only restriction on the voters' approval sets is that each voter approves of at least one candidate, just as is standard in the single-winner literature. However, whereas in the $k = 1$ special case this assumption is sufficient to ensure that a uniform cut-off utility lower bound for each voter is feasible, the same is not true for general k.

Definition 4.1 (IFS in ABC voting)**.** A fractional committee $\vec{p}$ is said to provide *individual fair share (IFS)* if for each voter $i \in N$,

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \frac{1}{n} \cdot \min\{k, |A_i|\}.$$

While IFS captures interpretation (a) of fair share, Strong IFS defined below reflects interpretation (b) which says that each voter should control $1/n$ of the fractional committee.

[5]For instance, let $k > n$ and consider the case where voter i only approves a single candidate. Then, the above inequality cannot hold for i as the left-hand side is upper bounded by $|A_i| = 1$ while the right-hand side is greater than one and can be arbitrarily large.

Definition 4.2 (Strong IFS in ABC voting)**.** A fractional committee $\vec{p}$ is said to provide *strong individual fair share (Strong IFS)* if for each voter $i \in N$,

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \min\left\{\frac{k}{n}, |A_i|\right\}.$$

Next, we strengthen IFS (resp., Strong IFS) to *unanimous fair share (UFS)* (resp., *Strong UFS*), which, at a high level, guarantees any group of like-minded voters an influence proportional to its size.

Definition 4.3 (UFS and Strong UFS in ABC voting)**.** A fractional committee $\vec{p}$ is said to provide

- *unanimous fair share (UFS)* if for any group of voters $S \subseteq N$ with $A_i = A_j$ for any pair of voters $i, j \in S$, then the following holds for each $i \in S$:

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \frac{|S|}{n} \cdot \min\{k, |A_i|\};$$

- *strong unanimous fair share (Strong UFS)* if for any group of voters $S \subseteq N$ with $A_i = A_j$ for any pair of voters $i, j \in S$, then the following holds for each $i \in S$:

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c \geq \min\left\{|S| \cdot \frac{k}{n}, |A_i|\right\}.$$

Bei et al. [2025] introduced the model of *cake sharing*: The cake is modelled as an interval $[0, c]$; for a given parameter $\alpha \in [0, c]$, a subset of the cake (termed as "an allocation") with length of at most α can be collectively allocated to n agents who are assumed to have *piecewise uniform* (i.e., binary) utilities over the cake. The ABC voting considered in this subsection corresponds to a special case of the cake-sharing model where $\alpha = k$ and for each $j \in C$, the interval $[j-1, j]$ represents candidate j.

Inspired by the EJR notion of Aziz et al. [2017] for the setting of ABC voting, Definition 7.2 of Bei et al. [2025] adapted it to the cake-sharing setting. To distinguish from the version of EJR introduced by Aziz et al. [2017], following the nomenclature of Lu et al. [2024], we refer to the version defined in the cake-sharing setting as *cake EJR*. Below, we present the definition of cake EJR in the context of ABC voting.

Definition 4.4 (Cake EJR in ABC voting)**.** A fractional committee $\vec{p}$ with $\sum_{c \in C} p_c \leq \alpha$ is said to satisfy *cake EJR* if for every positive real number t and every group of agents $N' \subseteq N$ such that $|N'| \geq t \cdot \frac{n}{\alpha}$ and $|\bigcap_{i \in N'} A_i| \geq t$, then it holds that $u_j(\vec{p}) \geq t$ for some agent $j \in N'$.

A fractional committee providing cake EJR always exists [Bei et al., 2025; Lu et al., 2024].

We now show that cake EJR implies Strong UFS (and therefore Strong IFS). Fix any unanimous group of voters $S \subseteq N$ such that $A_i = A_j$ for any pair of voters $i, j \in S$. It can be seen that as long as $t \leq |S| \cdot \frac{\alpha}{n}$, a cake EJR allocation also satisfies Strong UFS. The reverse direction does not hold, as cake EJR also requires representation to be guaranteed for groups of voters that are large enough and with sufficient intersection of their approvals.

Our next axiom — *group fair share (GFS)* — gives a non-trivial ex-ante representation guarantee to *every* coalition of voters. As we will see shortly, GFS and cake EJR are logically independent.

Definition 4.5 (GFS in ABC voting)**.** A fractional committee $\vec{p}$ is said to provide *group fair share (GFS)* if the following holds for every group of voters $S \subseteq N$:

$$\sum_{c \in \bigcup_{i \in S} A_i} p_c \geq \frac{1}{n} \cdot \sum_{i \in S} \min\{k, |A_i|\}.$$

We note that a GFS fractional committee always exists and can be computed by a very natural algorithm called *Random Dictator*, which selects each voter's favourite integral committee (breaking ties arbitrarily) with probability $1/n$.

Proposition 4.6. *Random Dictator computes a randomized committee that is ex-ante GFS in polynomial time.*

Proof. First, it is clear that Random Dictator runs in polynomial time. Let $\{(\frac{1}{n}, W_i)\}_{i \in N}$ be the randomized committee returned by Random Dictator for an instance of our problem, where W_i denotes voter i's favourite integral committee and is of size at most k. Let $\vec{p}$ be the fractional committee it implements. Note that $p_c = \sum_{i \in N} \frac{1}{n} \cdot \mathbb{1}_{\{c \in W_i\}}$ for all $c \in C$, where $\mathbb{1}_{\{c \in W_i\}}$ is an indicator function that is 1 if $c \in W_i$ and 0 otherwise.

Fix any $S \subseteq N$. Substituting to the LHS of the GFS guarantee (see Definition 4.5), we get

$$\begin{aligned}\sum_{c \in \bigcup_{i \in S} A_i} p_c &= \sum_{c \in \bigcup_{i \in S} A_i} \sum_{j \in N} \frac{1}{n} \cdot \mathbb{1}_{\{c \in W_j\}} \\ &\geq \frac{1}{n} \cdot \sum_{j \in S} \sum_{c \in \bigcup_{i \in S} A_i} \mathbb{1}_{\{c \in W_j\}} = \frac{1}{n} \cdot \sum_{j \in S} \left| W_j \cap \bigcup_{i \in S} A_i \right| \geq \frac{1}{n} \cdot \sum_{j \in S} \min\{k, |A_j|\},\end{aligned}$$

where the last transition holds because W_j is one of voter j's most preferred committees by the definition of Random Dictator. $\square$

However, Random Dictator does not satisfy Strong IFS.[6] Indeed, this is the principal reason we chose to name the respective axiom hierarchies as we did. There is a significant precedent of treating the Random Dictator as the utility lower bound for fair share axioms, including the work of Bogomolnaia et al. [2005, pp. 167] who introduced fair share notions:

> *Fair welfare share uses the random dictator mechanism as the disagreement option that each participant is entitled to enforce.*

Furthermore, the natural extensions of Strong UFS to "Strong GFS" are not guaranteed to exist. For instance, following the work of Bogomolnaia et al. [2005] and Brandl et al. [2021] as well as our own Definition 4.5, we may be tempted to formulate the RHS of Strong GFS as the sum of the Strong IFS guarantees, i.e., $\sum_{i \in S} \min\left\{\frac{k}{n}, |A_i|\right\}$. However, Example 4.7 will show that a fractional committee satisfying this notion may not always exist. Another natural generalization would be the following:

$$\sum_{c \in \bigcup_{i \in S} A_i} p_c \geq \min\left\{|S| \cdot \frac{k}{n}, \left|\bigcup_{i \in S} A_i\right|\right\} \tag{1}$$

Equation (1) captures the spirit of strong fair share well by affording each coalition of voters control over the outcome proportional to their size, upper bounded by the number of candidates they

[6] To see this, consider an instance with $k = 2$, three candidates $\{c_1, c_2, c_3\}$, and two voters with $A_1 = \{c_1\}$ and $A_2 = \{c_2, c_3\}$. Since each voter must select an integral committee, voter 1 allocates some of her probability to a candidate she does not approve, and thus $\sum_{c \in A_1} p_c = p_{c_1} = 1/2 < \min\left\{\frac{k}{n}, |A_1|\right\} = 1$.

collectively approve. Example 4.7 below shows the formulation of Strong GFS given by Equation (1) is also impossible to satisfy.

Example 4.7. *Consider an instance with $n = 4$, $k = 4$, and the following approval sets:*

$$A_1 = A_2 = \{c_1\} \qquad A_3 = \{c_1, c_2, c_3\} \qquad A_4 = \{c_1, c_4, c_5\}.$$

For the group $T = \{1, 2, 3\}$, Equation (1) requires that

$$\sum_{c \in \bigcup_{i \in T} A_i} p_c \geq \min\left\{ |T| \cdot \frac{k}{n}, \left| \bigcup_{i \in T} A_i \right| \right\} = 3.$$

This means that each candidate in A_3 must receive probability 1. By symmetry, the same holds for the group $\{1, 2, 4\}$ and thus A_4. However, since $|A_3 \cup A_4| = 5$ and $k = 4$, this is an impossibility.

Suzuki and Vollen [2024] introduced in the setting of ABC voting a novel ex-ante proportionality axiom called *group resource proportionality (GRP)* that is stronger than both Strong UFS and GFS. For the sake of being self-contained, we provide below the definition of GRP.

Definition 4.8 (GRP in ABC voting)**.** A fractional committee $\vec{p}$ is said to provide *group resource proportionality (GRP)* if for every $S \subseteq N$:

$$\sum_{c \in \bigcup_{i \in S} A_i} p_c \geq |S| \cdot \frac{k}{n} - \max_{T \subseteq S} \left[|T| \cdot \frac{k}{n} - |\bigcup_{i \in T} A_i| \right].$$

It follows directly from the definitions that GFS implies UFS, which in turn implies IFS, and that each of our generalizations of IFS, UFS, and GFS correspond to their definitions in the single-winner voting scenarios. The relations between these axioms as well as cake EJR and GRP are pictured in Figure 1. We would also like to remark that any pair of ex-ante fairness notions not given explicit logical relation in Figure 1 are logically independent (i.e., neither implies the other). The details are deferred to Appendix A.

4.2 Fairness for Fractional PB Outcomes

We are now ready to generalize the (strong) fair share axiom hierarchies to the general PB setting, where projects have heterogeneous costs and voters have general additive utilities, with the intention of formulating axioms which (i) collapse to those defined in Section 4.1 for approval-based committee voting, and (ii) reflect their respective interpretations as detailed below:

(a) Fair share: each voter is given $\frac{1}{n}$ probability to choose their favourite *fractional* outcome;

(b) Strong fair share: each voter can select $\frac{1}{n}$ of the fractional outcome.

We begin with *individual fair share (IFS)*, which guarantees each agent a utility of at least a $\frac{1}{n}$-fraction of the utility they receive from their favourite fractional outcome.

Definition 4.9 (IFS in PB)**.** A fractional outcome $\vec{p}$ is said to provide *IFS* if for each $i \in N$,

$$u_i(\vec{p}) \geq \frac{1}{n} \cdot \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}).$$

The quantity expressed by the max-operator is the utility-maximizing fractional outcome for the agent i, and hence it is immediately clear that Definition 4.9 follows interpretation (a). In general, this can be computed by selecting projects in order of descending utility per unit cost. For binary utilities, this means selecting approved projects in order of ascending cost. We can already observe that, in the PB setting, IFS (Definition 4.9) seems quite a bit more demanding than its ABC voting counterpart (Definition 4.1), in which a project/candidate can only take on a utility per unit cost value of 1 or 0. In contrast, in the PB setting, each voter-project pair could result in a unique utility per unit cost value.

For Strong IFS, keeping with interpretation (b) above, an agent's utility lower bound is given by the optimal utility they could achieve if given their proportion of the budget.

Definition 4.10 (Strong IFS in PB)**.** A fractional outcome $\vec{p}$ is said to provide *Strong IFS* if for each $i \in N$,

$$u_i(\vec{p}) \geq \max_{\vec{t} \in \mathcal{X}\left(\frac{B}{n}\right)} u_i(\vec{t}).$$

Next, we strengthen IFS (resp., Strong IFS) to *unanimous fair share (UFS)* (resp., *Strong UFS*), which strengthens the fair share utility guarantee for groups of voters with identical preferences.

Definition 4.11 (UFS and Strong UFS in PB)**.** A fractional outcome $\vec{p}$ is said to provide

- *UFS* if for any $S \subseteq N$ where $u_i = u_j$ for any $i, j \in S$, then the following holds for each $i \in S$:

$$u_i(\vec{p}) \geq \frac{|S|}{n} \cdot \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}).$$

- *Strong UFS* if for any $S \subseteq N$ where $u_i = u_j$ for any $i, j \in S$, the following holds for each $i \in S$:

$$u_i(\vec{p}) \geq \max_{\vec{t} \in \mathcal{X}\left(|S| \cdot \frac{B}{n}\right)} u_i(\vec{t}). \tag{2}$$

As its name suggests, Strong UFS implies UFS (and similarly, Strong IFS implies IFS).[7] While (Strong) UFS gives a utility guarantee to groups of voters with identical preferences, our next axiom — *group fair share (GFS)* — extends a non-trivial representation guarantee to all groups of voters.

Definition 4.12 (GFS in PB)**.** A fractional outcome $\vec{p}$ is said to provide *GFS* if the following holds for any $S \subseteq N$:

$$\sum_{j \in C} \left(p_j \cdot \max_{i \in S} u_i(j) \right) \geq \frac{1}{n} \cdot \sum_{i \in S} \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}).$$

In committee voting, the LHS of GFS is simply the probability allocated to candidates in the union of the group of voters' approval sets. Thus, while it is clear that Definition 4.12 collapses to the GFS in committee voting (Definition 4.5), this definition is not the only formulation of the LHS of GFS which does so. For example, instead of taking the maximum utility for each project j over all agents in S, we could have instead taken the average or median (or minimum) utility over all agents in S with non-zero utility for project j. Of these options, our formulation results in the weakest definition. Since, as we will see, this definition of GFS is not compatible with any of the ex-post fairness notions we consider in most PB settings, each of the impossibility results considering GFS in this paper would also hold for any stronger variant of GFS.

[7] To see this, let $\vec{q} = \arg\max_{\vec{t} \in \mathcal{X}} u_i(\vec{t})$. Now simply note that $\frac{|S|}{n} \cdot \vec{q} \in \mathcal{X}\left(|S| \cdot \frac{B}{n}\right)$.

Fractional Random Dictator We now extend the well-known Random Dictator algorithm [Bogomolnaia et al., 2005] to the computation of fractional PB outcomes. The high-level idea of the algorithm is to compute the fractional outcome resulting from giving each voter $1/n$ probability to select their own favourite fractional outcome. For a voter $i \in N$, let X_i be the maximal set of projects which can be funded fully in order of maximum utility per unit cost and let g_i be the project with highest utility per unit cost in the remaining set of projects. Also, denote the indicator function as $\mathbb{1}_{\{\cdot\}}$. For each $j \in C$, the *Fractional Random Dictator* algorithm outputs the fractional outcome $\vec{p}$ defined as follows:

$$p_j = \frac{1}{n} \cdot \sum_{i \in N} \left(\mathbb{1}_{\{j \in X_i\}} + \mathbb{1}_{\{j = g_i\}} \cdot \frac{B - \text{cost}(X_i)}{\text{cost}(j)} \right).$$

It can be verified easily that this Fractional Random Dictator algorithm degenerates to the Random Dictator algorithm for ABC voting described in Section 4.1. We show below an ex-ante GFS fractional outcome always exists, and can be computed via Fractional Random Dictator. Its proof is deferred to Appendix B.

Theorem 4.13. *Fractional Random Dictator computes an ex-ante GFS fractional outcome in polynomial time.*

5 BoBW Fairness under Cost Utilities

We start by investigating in Section 5.1 the logical relations between our ex-ante and ex-post properties for *integral* outcomes in the setting of approval-based committee (ABC) voting. In doing so, we rule out some naive approaches to our problem of interest and illustrate the usefulness of our ex-ante properties. Next, in Section 5.2, we turn our attention to "PB with cost utilities," a setting generalizing ABC voting.

5.1 BoBW Fairness in Approval-Based Committee Voting

In this subsection, we are concerned with the setting of approval-based committee (ABC) voting. We begin by remarking that there may not exist an integral committee satisfying positive share, let alone any other stronger ex-ante fairness properties defined in Section 4.1 — this is the principal motivation for studying randomized committees. Nevertheless, we seek to understand what our fairness concepts for fractional committees can tell us about the space of integral committees satisfying the ex-post fairness properties of interest (see, e.g., Definition 2.1). The following example shows that our fractional fairness concepts help in reasoning about which outcomes satisfying our ex-post properties are more desirable.

Example 5.1. *Consider an instance with $n = 10$ and $k = 4$. Suppose eight of the voters approve of candidates $\{c_1, c_2, c_3, c_4\}$ and the remaining two voters approve of candidate $\{c_5\}$.*

Note that the committee $W = \{c_1, c_2, c_3, c_4\}$ satisfies EJR. This is because $4 \cdot \frac{10}{4} > 8 \geq 3 \cdot \frac{10}{4}$ and W already includes at least three candidates approved by the eight voters. Also, since $2 < 1 \cdot \frac{10}{4}$, EJR does not guarantee the two voters who approve $\{c_5\}$ being represented in W, violating positive share.

The alternative committee of $\{c_1, c_2, c_3, c_5\}$ also satisfies EJR, and additionally satisfies IFS.

As shown by Example 5.1, even when an integral committee satisfying IFS exists, some EJR outcomes may not satisfy positive share. From this, we conclude that a successful algorithm must select carefully from the space of outcomes satisfying our ex-post properties. We next explore to what extent our ex-ante properties imply our ex-post properties in the integral case.

Proposition 5.2. *If an integral committee satisfies IFS, then it satisfies JR.*

Proof. Let W be an integral committee which satisfies IFS and $\vec{p} = \vec{1}_W$. Then, for all $i \in N$, we have

$$u_i(\vec{p}) = \sum_{c \in A_i} p_c = |A_i \cap W| \geq \frac{\min(|A_i|, k)}{n} > 0.$$

Thus, since $|A_i \cap W|$ is an integer, $|A_i \cap W| \geq 1$ for all $i \in N$ and it follows that W is JR. □

While Proposition 5.2 hints at a synergy between our ex-ante and ex-post properties, Propositions 5.3 and 5.4 below show that even the strongest ex-ante properties in our hierarchy do not imply the next strongest ex-post property.

Proposition 5.3. *If an integral committee satisfies Strong UFS, it does not necessarily satisfy PJR.*

Proof. Consider an instance with $k = 3$ and $n = 4$ and the following approval profile:

$$A_1 = \{c_1, c_6, c_7\} \qquad A_2 = \{c_2, c_6, c_7\} \qquad A_3 = \{c_3, c_6, c_7\} \qquad A_4 = \{c_4, c_5, c_6\}.$$

The committee $W = \{c_4, c_5, c_6\}$ satisfies Strong UFS since, for each voter $i \in [3]$, $|W \cap A_i| = 1 \geq \min\left\{|A_i| \cdot \frac{k}{n}, |A_i|\right\} = \min\left\{1 \cdot \frac{3}{4}, 3\right\}$, and voter 4 receives their most preferred committee. Note that Strong UFS does not apply to any group of voters of cardinality size at least 2 as the voters have diverse approval sets. Now see that voters $S = [3]$ forms a 2-cohesive group as $|S| = 3 \geq 2 \cdot \frac{n}{k} = 2 \cdot \frac{4}{3}$ and they commonly approve two candidates. The committee, however, fails PJR as $|W \cap \bigcup_{i \in S} A_i| = |\{c_6\}| = 1 < 2$. □

Proposition 5.4. *If an integral committee satisfies GFS, it does not necessarily satisfy PJR.*

Proof. Consider an instance with $k = 4$ and $n = 2$ and the following approval profile:

$$A_1 = \{c_1, c_2\} \qquad A_2 = \{c_2, c_3, c_4, c_5\}.$$

The committee $W = \{c_2, c_3, c_4, c_5\}$ satisfies GFS since $|W \cap A_1| = 1 = \frac{1}{n} \cdot \min\{k, |A_1|\}$ (and voter 2 receives their most preferred committee). Now see that voter $\{1\}$ forms a 2-cohesive group; however, $|A_1 \cap W| < 2$, meaning that W does not satisfy PJR. □

Despite the negative findings, we proved the following compatibility results in the work by Aziz et al. [2023b]:

- ex-ante GFS, ex-ante Strong UFS and ex-post EJR;
- ex-ante Strong UFS and ex-post FJR.

We left open the compatibility between ex-ante GFS, ex-ante Strong UFS and ex-post FJR. Later, Suzuki and Vollen [2024] resolved this open question in the affirmative by showing that ex-post FJR can be achieved alongside ex-ante GRP (Definition 4.8), a notion that implies both ex-ante GFS and ex-ante Strong UFS.[8]

Theorem 5.5 (Suzuki and Vollen, 2024, Corollary 5.4)**.** *In approval-based committee voting, a randomized committee satisfying ex-post FJR and ex-ante GRP (therefore, ex-ante GFS and ex-ante Strong UFS) is guaranteed to exist.*

[8] In contrast to our Propositions 5.3 and 5.4, if an integral committee satisfies GRP, then it satisfies PJR [Suzuki and Vollen, 2024, Proposition 3.7].

Algorithm 1: Cake EJR, Ex-ante GRP & Ex-post EJR in Approval-Based Committee Voting

Input: Voters $N = [n]$, candidates $C = [m]$, approvals $(A_i)_{i \in N}$, and committee size k

Output: A cake EJR, GRP fractional committee $\vec{p} = (p_c)_{c \in C}$ and its implementation as a lottery over integreal EJR committees.

1 Initialize $N' \leftarrow N$, $W \leftarrow \emptyset$ and m-dimensional vector $\vec{q} \leftarrow (0, 0, \ldots, 0)$.

2 **while** $N' \neq \emptyset$ **do**

3   Let t^* be the largest non-negative real number such that there exists a t^*-cohesive group $N^* \subseteq N'$, i.e., $|N^*| \geq t^* \cdot \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq t^*$, breaking ties in favour of larger $|N^*|$.

4     $B \leftarrow$ a set of $\lfloor t^* \rfloor$ arbitrary candidates from $\bigcap_{i \in N^*} A_i$

5   **if** *t^* is not an integer* **then**

6         $c' \leftarrow$ an arbitrary candidate from $(\bigcap_{i \in N^*} A_i) \setminus B$

7         $q_{c'} \leftarrow \min\{q_{c'} + (t^* - |B|), 1\}$

8   **foreach** $b \in B$ **do** $q_b \leftarrow 1$

9     $W \leftarrow W \cup B$

10     $N' \leftarrow N' \setminus N^*$

11 Apply Lemma 6.1 of Suzuki and Vollen [2024] to $\vec{q}$ which yields a GRP committee $\vec{p}$ such that $p_c \geq q_c$ for each $c \in C$.

12 Increase p_c, with each being capped at 1, such that $\sum_{c \in C} p_c = k$.

13 Apply a randomized rounding scheme [e.g., Aziz et al., 2019; Gandhi et al., 2006] to $\vec{p}$, which outputs a lottery over integral committees of size k. Let $\{(\lambda_j, W_j)\}_{j \in [s]}$ denote the randomized committee.

14 **return** $\vec{p}$ *and its implementation* $\{(\lambda_j, W_j)\}_{j \in [s]}$

This result came as a corollary to a more general result which used flow networks to show that any *affordable* committee could be completed to a fractional committee which satisfies GRP. In the following subsection, we will extend the approach taken by Suzuki and Vollen [2024] to the more general setting of PB with Cost Utilities and establish an analog of their result.

We conclude this subsection by remarking that GRP provides a non-trivial ex-ante representation guarantee to every coalition of voters, but, by itself, it does not guarantee proportional representation to cohesive groups as does cake EJR — this can be seen from Example A.3. Nevertheless, we show below that these two ex-ante fairness notions are compatible and, moreover, can be achieved alongside strong ex-post proportional representation guarantees.

Theorem 5.6. *In approval-based committee voting, a randomized committee satisfying ex-ante cake EJR, ex-ante GRP and ex-post EJR is guaranteed to exist.*

Proof. We will show Algorithm 1 outputs a fractional committee $\vec{p}$ that satisfies both cake EJR and ex-ante GRP and, moreover, can be implemented by a lottery over integral EJR committees.

Initialize $\vec{q} \leftarrow (0, 0, \ldots, 0)$ in line 1 as a fractional committee. In each iteration of the `while`-loop, with the current voters N', Algorithm 1 identifies the largest *non-negative real number* t^* such that there exists a t^*-cohesive group $N^* \subseteq N'$, i.e., $|N^*| \geq t^* \cdot \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq t^*$. Our algorithm then constructs a fractional committee $\vec{q}$ with t^* candidates that are approved by each member of the cohesive group. Note that we include $\lfloor t^* \rfloor$ candidates in their entirety (line 4) and one candidate of fraction at least $t^* - \lfloor t^* \rfloor$ (see the `if`-statements). It is worth noting that the `while`-loop procedure is an application of the GreedyEJR-M rule formulated in the paper of Lu et al. [2024] to our present setting. Due to Theorem 3.8 of Lu et al. [2024], we thus have that the `while`-loop always terminates, and produces a fractional committee $\vec{q}$ with $\sum_{c\in C} q_c \leq k$ satisfying cake EJR.

We now show that the set of candidates W, which is constructed in line 9 and each member of which is integrally selected, satisfies EJR. Note that any t^*-cohesive group of voters N^* considered by Algorithm 1 also form a $\lfloor t^* \rfloor$-cohesive group, and Algorithm 1 will thus include $\lfloor t^* \rfloor$ candidates $B \subseteq \bigcap_{i\in N^*} A_i$ in their entirety in line 4. Assume for contradiction that for some group X, EJR fails for X and *integer* t. Consider the moment after the `while`-loop removed the last group with parameter $t^* \geq t$. If no voter in X has been removed, the `while`-loop should have removed X with parameter t, a contradiction. Otherwise, some voter $j \in X$ has been removed. In this case, the `while`-loop guarantees to include at least t candidates in their entirety, which means that EJR is satisfied for group X with integer t, a contradiction. It follows that the integral committee W satisfies EJR.

After the `while`-loop, in line 11, Algorithm 1 applies Lemma 6.1 of Suzuki and Vollen [2024], which states that any fractional committee representable as a feasible flow on their network can be extended to a GRP committee. To apply this lemma, we need to show that $\vec{q}$ can be represented as a feasible flow. This requires demonstrating that fractional committee $\vec{q}$ can be paid for by the voters such that no voter exceeds their budget of $\frac{k}{n}$ and that voters only pay for candidates they approve. We present an explicit payment scheme to show that this is indeed possible. In particular, we show that in each iteration of the `while`-loop, the voters in N^* can pay for the candidates who receive additional shares, without exceeding their budgets and only paying for candidates they approve. Since the sets of voters encountered in each iteration are disjoint, this suffices to construct the desired payment scheme.

In each iteration, the voters in N^* have a collective budget of $|N^*| \cdot \frac{k}{n} \geq t^*$. The set of candidates receiving additional shares lies in $\bigcap_{i\in N^*} A_i$, so by construction, voters only pay for candidates they approve. Let B denote the integrally selected candidates with $|B| = \lfloor t^* \rfloor$. For each $b \in B$, we charge each voter in N^* an amount of at most $\frac{1}{|N^*|}$, so that each voter spends a total of at most $\frac{|B|}{|N^*|} = \frac{\lfloor t^* \rfloor}{|N^*|}$ on candidates in B. For the one additional candidate $c' \in (\bigcap_{i\in N^*} A_i) \setminus B$ that receives the remaining fractional share of at most $t^* - \lfloor t^* \rfloor$, this amount is charged uniformly among the voters in N^*, so each voter contributes at most $\frac{t^* - \lfloor t^* \rfloor}{|N^*|}$. Thus, each voter's total spending is bounded by

$$\frac{\lfloor t^* \rfloor}{|N^*|} + \frac{t^* - \lfloor t^* \rfloor}{|N^*|} \leq \frac{t^*}{|N^*|} \leq \frac{k}{n}.$$

This confirms that the desired payment scheme exists, allowing us to apply Lemma 6.1 of Suzuki and Vollen [2024]. It follows that $\vec{p}$ satisfies GRP.

Finally, since $p_c \geq q_c$ for each $c \in C$, $\vec{p}$ satisfies cake EJR, as $\vec{q}$ does. Moreover, as it is in $\vec{q}$, committee W is selected integrally in $\vec{p}$. By the first property of Gandhi et al. [2006, Theorem 2.3], W is included in every realization. Consequently, the output lottery satisfies ex-post EJR. □

5.2 BoBW Fairness in PB with Cost Utilities

In this subsection, we consider the model restriction in which voters have *cost utilities* [Talmon and Faliszewski, 2019], that is, $u_i(c) \in \{0, \text{cost}(c)\}$ for all voters $i \in N$ and projects $c \in C$. We refer to this setting as *PB with cost utilities*, which generalizes the setting of ABC voting discussed in the last subsection. We point out that approval sets A_i's are still well-defined in this setting as the set of projects $c \in C$ for which $u_i(c) \neq 0$. This setting has been studied in a number of papers on participatory budgeting [see, e.g., Rey et al., 2025, for an overview] and is sometimes referred

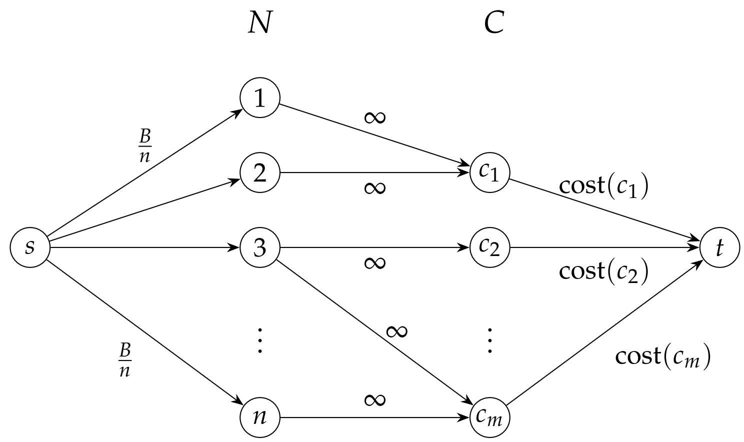

Figure 3: Illustration of the network representation of an instance of PB with Cost Utilities.

to as "cost-based satisfaction functions" (in contrast to "cardinality-based satisfaction functions" which we refer to as "binary utilities" in Section 6 of this work).

We start by extending the ex-ante notion known as *group resource proportionality (GRP)* from committee voting [Suzuki and Vollen, 2024] to our PB setting with cost utilities.

Definition 5.7 (GRP in PB with cost utilities)**.** A fractional outcome $\vec{p}$ is said to provide *group resource proportionality (GRP)* if for every $S \subseteq N$:

$$\sum_{c \in \bigcup_{i \in S} A_i} (p_c \cdot \text{cost}(c)) \geq |S| \cdot \frac{B}{n} - \max_{T \subseteq S} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right].$$

One benefit of GRP is that it offers a natural flow network interpretation of PB with cost utilities, wherein each voter is given a budget of $\frac{B}{n}$ and can flow their budget towards covering the costs of only those projects which they approve. Below, we give a formal definition of this network formulation, which, in coarse terms, connects a source to voters, voters to their approval sets, and candidates to a sink; see Figure 3 for an illustration.

Definition 5.8. The *network representation* $\mathcal{N}$ of an instance of PB with cost utilities is a flow network with source s, sink t, a node for each voter $i \in N$ and each candidate $c \in C$, and capacities for all network arcs defined as follows:

- $\text{cap}(s, i) = B/n$ $\quad\quad \forall\, i \in N;$
- $\text{cap}(i, c) = \infty$ $\quad\quad \forall\, i \in N, c \in A_i;$
- $\text{cap}(c, t) = \text{cost}(c)$ $\quad\quad \forall\, c \in C.$

We now characterize fractional outcomes satisfying GRP using the network formulation of PB with cost utilities, specifically showing a correspondence with max flows on the network. We note that its proof resembles the argument of the GRP characterization result in ABC voting [Suzuki and Vollen, 2024], and therefore defer the proof to Appendix B.

Theorem 5.9. *Given an instance of PB with cost utilities, a fractional outcome $\vec{p}$ of the instance satisfies GRP if and only if there exists a maximum flow f on the network representation $\mathcal{N}$ such that $p_c \cdot \text{cost}(c) \geq f(c, t)$ for all $c \in C$.*

In the committee voting setting, GRP implies both GFS and Strong UFS [Suzuki and Vollen, 2024]. This also holds for our generalization of GRP to PB with cost utilities.

Proposition 5.10. *In PB with cost utilities, GRP implies Strong UFS and GFS.*

Proof. We first show that GRP implies Strong UFS. Suppose $S \subseteq N$ is some group of voters with identical approval sets, i.e., $A_i = A_j$ for all $i, j \in S$. Given some fractional outcome $\vec{p}$ satisfying GRP, it holds for each voter $i \in S$ that:

$$\begin{aligned} u_i(\vec{p}) = \sum_{c \in A_i} (p_c \cdot \text{cost}(c)) &= \sum_{c \in \bigcup_{i \in S} A_i} (p_c \cdot \text{cost}(c)) \\ &\geq |S| \cdot \frac{B}{n} - \max_{T \subseteq S} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{j \in T} A_j) \right] \\ &= |S| \cdot \frac{B}{n} - \max_{T \in \{\emptyset, S\}} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{j \in T} A_j) \right] \\ &= |S| \cdot \frac{B}{n} - \max \left\{ |S| \cdot \frac{B}{n} - \text{cost}(A_i),\ 0 \right\} \\ &= \min \left\{ |S| \cdot \frac{B}{n},\ \text{cost}(A_i) \right\} \geq \max_{\vec{t} \in \mathcal{X}\left(|S| \cdot \frac{B}{n}\right)} u_i(\vec{t}), \end{aligned}$$

where the transition in the third line above holds because $\bigcup_{j \in T} A_j = A_i$ for any $T \neq \emptyset$ and thus the expression is maximized by setting $T = S$ if $T \neq \emptyset$.

Now, we will show that GRP implies GFS. Fix an arbitrary set of voters $S \subseteq N$. Let $Q = \{i \in S \mid \text{cost}(A_i) \leq B\}$ be the set of voters in S whose approval set costs no more than the budget. Let $T^* = \arg\max_{T \subseteq S} \left[|T| \cdot \frac{B}{n} - \text{cost}(\bigcup_{i \in T} A_i) \right]$ be the set of voters in S for which the maximum is attained in the right-hand side of GRP. Note that the empty set attains 0 in the expression T^* maximizes. This implies that if $T^* \neq \emptyset$, then every voter in T^* must belong to Q, since otherwise any voter $i \in T^*$ but $i \notin Q$ would make the expression T^* maximizes be negative. Again, taking $\vec{p}$ to be some fractional outcome satisfying GRP, we see:

$$\begin{aligned} \sum_{j \in C} (p_j \cdot \max_{i \in S} u_i(j)) = \sum_{c \in \bigcup_{i \in S} A_i} (p_c \cdot \text{cost}(c)) &\geq |S| \cdot \frac{B}{n} - \max_{T \subseteq S} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \\ &= |S| \cdot \frac{B}{n} - \max_{T \subseteq Q} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \\ &\geq |S| \cdot \frac{B}{n} - \max_{T \subseteq Q} \left[|T| \cdot \frac{B}{n} - \frac{1}{n} \cdot \sum_{i \in T} \text{cost}(A_i) \right] \\ &= |S| \cdot \frac{B}{n} - \left[|Q| \cdot \frac{B}{n} - \frac{1}{n} \cdot \sum_{i \in Q} \text{cost}(A_i) \right] \\ &= |S \setminus Q| \cdot \frac{B}{n} + \frac{1}{n} \cdot \sum_{i \in Q} \text{cost}(A_i) \\ &= \frac{1}{n} \cdot \sum_{i \in S} \min(B,\ \text{cost}(A_i)) = \frac{1}{n} \cdot \sum_{i \in S} \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}). \end{aligned}$$

The transition in the third line follows from $\frac{1}{n} \cdot \sum_{i \in T} \text{cost}(A_i) \leq \frac{1}{n} \cdot \sum_{i \in T} \text{cost}(\bigcup_{j \in T} A_j) \leq \frac{|T|}{n} \cdot \text{cost}(\bigcup_{i \in T} A_i) \leq \text{cost}(\bigcup_{i \in T} A_i)$. The transition in the fourth line is due to the fact that the maximum is attained by Q, since each voter $i \in Q$ adds $\left(\frac{B}{n} - \frac{1}{n} \cdot \text{cost}(A_i)\right)$ to the expression, which is non-negative for each voter in Q by definition. $\square$

As we will show in Sections 6 and 7, best-of-both-worlds results involving GFS are a non-starter in the other PB settings we consider in this work. Since any faithful generalization of GRP would strengthen GFS, and our focus is on best-of-both-worlds results, we restrict our treatment of GRP to the present setting. Toward establishing a general best-of-both-worlds result in the PB setting with cost utilities, we now define an ex-post property on integral outcomes known as *affordability*, which weakens *priceability* [Peters et al., 2021] and was first defined for committee voting by Brill and Peters [2024].

Definition 5.11 (Affordability)**.** An outcome W is said to satisfy *affordability* (alternatively, W is *affordable*) if there exists a payment function $\pi_i \colon C \to \mathbb{R}_{\geq 0}$ specifying the amount of money voter i pays for each project, which satisfies the following conditions:

1. $\pi_i(c) = 0$ for each $i \in N$ and $c \notin A_i$;
2. $\sum_{c \in C} \pi_i(c) \leq B/n$ for each $i \in N$;
3. $\sum_{i \in N} \pi_i(c) = \mathrm{cost}(c)$ for each $c \in W$;
4. $\sum_{i \in N} \pi_i(c) = 0$ for each $c \notin W$.

Intuitively, an outcome is affordable if, after dividing the total budget B evenly amongst all voters, it is possible to cover the costs of the selected projects exactly without requiring any voter to pay for a project which they do not approve. By requiring that voters only use their virtual budgets toward projects which they approve, affordability can be seen as a fairness criterion. In fact, the central theorem of this section shows that affordability is sufficient to guarantee that an integral outcome can be "completed" to a fractional outcome which satisfies GRP.

Theorem 5.12. *Let W be an affordable outcome given any instance of PB with cost utilities. There exists a polynomial-time algorithm which computes an integral outcome sampled from a lottery $\{(\lambda_j, W_j)\}_{j \in [s]}$ that is ex-ante GRP and ex-post BB1, and it holds that $W_j \supseteq W$ for all $j \in [s]$.*

Proof. Denote by $\{\pi_i\}_{i \in N}$ a set of payment functions satisfying Conditions 1–4 for W in Definition 5.11. We build a flow network $\mathcal{N}$ to represent our instance following Definition 5.8 and transform the payment functions π_i's into a feasible flow on this network. We now define a flow f and show that f is feasible on $\mathcal{N}$:

- $f(s, i) = \sum_{c \in C} \pi_i(c)$ for all $i \in N$;
- $f(i, c) = \pi_i(c)$ for all $i \in N, c \in C$;
- $f(c, t) = \sum_{i \in N} \pi_i(c)$ for all $c \in C$.

It can be verified that f respects conservation of flows. Observe that Condition 2 of Definition 5.11 ensures that $f(s, i) \leq B/n = \mathrm{cap}(s, i)$ for all $i \in N$, Condition 1 ensures that $f(i, c) = 0 = \mathrm{cap}(i, c)$ for all $i \in N$ and $c \notin A_i$, Condition 3 ensures that $f(c, t) = \mathrm{cost}(c) = \mathrm{cap}(c, t)$ for all $c \in W$ and Condition 4 ensures that $f(c, t) = 0 < \mathrm{cap}(c, t)$ for all $c \notin W$, and thus f does not violate any capacity constraint in $\mathcal{N}$.

We point out that one can compute in polynomial time another flow f^* which is a maximum flow on $\mathcal{N}$ and where $f^*(c, t) \geq f(c, t)$ for all $c \in C$. Moreover, $f^*(c, t) = f(c, t)$ for all $c \in W$. This can be done by iteratively sending flow along augmenting paths, which will lead to a maximum flow in polynomial time, and never decrease flows along any arc entering the sink.

Now consider the fractional committee $\vec{p}$ given by setting $p_c = \frac{f^*(c,t)}{\text{cost}(c)}$ for each $c \in C$. By Theorem 3.2, we can compute in polynomial time an integral outcome sampled from an ex-post BB1 lottery $\Delta = \{(\lambda_j, W_j)\}_{j\in[s]}$ which implements $\vec{p}$. Since f^* is a maximum flow on $\mathcal{N}$, we have by Theorem 5.9 that $\vec{p}$ and thus the lottery Δ satisfies ex-ante GRP. By Condition 3 of Definition 5.11, it holds for all $c \in W$ that

$$p_c = \frac{f^*(c,t)}{\text{cost}(c)} = \frac{f(c,t)}{\text{cost}(c)} = \frac{1}{\text{cost}(c)} \cdot \sum_{i\in N} \pi_i(c) = 1,$$

and it thus follows that $W \subseteq W_j$ for all $j \in [s]$. □

Theorem 5.12 demonstrates that any ex-post property which is compatible with affordability can be achieved in tandem with ex-ante GRP using an ex-post BB1 lottery. Peters et al. [2021] defined the (computationally intractable) *Greedy Cohesive Rule (GCR)*, which satisfies FJR, even in general PB. The outcome returned by GCR satisfies affordability [Peters et al., 2021, Lemma 2], leading us to the following corollary of Theorem 5.12.

Corollary 5.13. *In PB with cost utilities, a lottery satisfying ex-ante GRP (therefore, ex-ante Strong UFS and ex-ante GFS), ex-post BB1, and ex-post FJR is guaranteed to exist.*

The polynomial-time *Method of Equal Shares (MES)* of Peters et al. [2021] outputs affordable outcomes by construction, and is defined for even the most general PB setting with additive utilities. While MES satisfies EJR in ABC voting, it no longer satisfies EJR in PB with cost utilities.[9] Indeed, there is no polynomial-time algorithm that, given an instance of PB with cost utilities as input, always computes an outcome satisfying EJR, unless $\mathsf{P} = \mathsf{NP}$ [Brill et al., 2023, Theorem 3.3]. Nevertheless, MES obtains an "up to any" relaxation of EJR in the present setting [Brill et al., 2023], which we will define now. Given a set $T \subseteq C$ of projects, a subset $S \subseteq N$ of voters is said to be T-cohesive if $|S| \cdot \frac{B}{n} \geq \text{cost}(T)$ and $T \subseteq \bigcap_{i\in S} A_i$. An outcome W is said to satisfy *EJR up to any project (EJR-x)* if for each $T \subseteq C$ and every T-cohesive group of voters $S \subseteq N$, there is a voter $i \in S$ such that $u_i(W \cup \{c\}) > u_i(T)$ for all $c \in T \setminus W$. Brill et al. [2023, Theorem 3.5] shows that MES achieves EJR-x under cost utilities. Thus, we have the following corollary to Theorem 5.12.

Corollary 5.14. *In PB with cost utilities, an integral outcome can be sampled in polynomial time from a lottery that is ex-ante GRP (therefore, ex-ante Strong UFS and ex-ante GFS), ex-post BB1, and ex-post EJR-x.*

6 BoBW Fairness in PB with Binary Utilities

In this section, we consider the setting of PB with binary utilities, that is, the setting in which the voters have binary utilities and the projects have arbitrary costs. Our main focus is to investigate whether the ex-ante fair share notions defined in Section 4.2 can be achieved simultaneously with ex-post fairness properties based on justified representation (see Section 2.1). The remainder of this section is organized as follows:

- In Section 6.1, we show that it is impossible to simultaneously achieve ex-ante GFS and ex-post JR.

[9] This can be seen from the instance with budget $B = 2$, $n = 2$ voters, three projects c_1, c_2, c_3, and the following approval profile: $A_1 = \{c_1, c_3\}$ and $A_2 = \{c_2, c_3\}$. The costs of the three projects are $\text{cost}(c_1) = 1$, $\text{cost}(c_2) = 1$ and $\text{cost}(c_3) = 0.9$. MES first selects project c_3 and then terminates as neither agent has enough budget left to buy the project that is only approved by her. However, $S = \{1\}$ and $T = \{c_1\}$ witness the violation of EJR.

- In Section 6.2, we show constructively that ex-ante Strong UFS and ex-post FJR are compatible, though our randomized algorithm is not polynomial time.
- In Section 6.3, we devise a polynomial-time randomized algorithm which simultaneously achieves ex-ante Strong UFS and ex-post EJR.

6.1 Impossibility: Ex-ante GFS + Ex-post JR

Our first main result in PB with binary utilities states that it is impossible to simultaneously achieve ex-ante GFS and ex-post JR. Note that in the more restricted setting with unit-cost projects (i.e., approval-based committee voting), we discuss in Section 5.1 that ex-ante GFS is compatible even with ex-post FJR. Our impossibility result demonstrates a clear and strong separation between PB with binary utilities and approval-based committee voting.

Theorem 6.1. *In PB with binary utilities, ex-ante GFS and ex-post JR are incompatible.*

Proof. Consider an instance with $n \geq 6$ and the following approval sets and project costs:

- each voter $i \in N$ approves $A_i = \{g^*, a_i, b_i, c_i\}$ with $\text{cost}(g^*) = \frac{B}{2}$ and $\text{cost}(a_i) = \text{cost}(b_i) = \text{cost}(c_i) = \frac{B}{2} - \varepsilon$, where $\varepsilon < \frac{B}{2} - \frac{2B}{n}$;
- note that g^* is the common project approved by every voter, and for any pair of voters $i \neq j$, $a_i, b_i, c_i \notin A_j$.

We establish the incompatibility using this instance by showing that *any* feasible fractional outcome satisfying GFS cannot be implemented by *any* lottery that is ex-post JR, even without imposing BB1. Suppose, for the sake of contradiction, that $\{(\lambda_j, W_j)\}_{j\in[s]}$ is an ex-post JR lottery implementing GFS fractional outcome $\vec{p}$.

We first point out that some integral outcome in the lottery includes g^*, and hence $p_{g^*} > 0$.

Claim 6.2. *There exists an outcome W_j such that $g^* \in W_j$.*

Proof. Suppose for the sake of contradiction that every integral outcome does not contain g^*. Fix any outcome W_j. Consider the set of voters $N' = \{i \in N \mid A_i \cap W_j = \emptyset\}$. Recall that for any pair of voters $i \neq i'$, $a_i, b_i, c_i \notin A_{i'}$. If $|W_j| < n/2$, then $|N'| \geq n/2$. It means that W_j is not JR because every voter in the $\{g^*\}$-cohesive group N' gets zero utility. Therefore, $|W_j| \geq n/2$. Since $g^* \notin W_j$ and every other project has an identical cost of $\frac{B}{2} - \varepsilon$, we have $\text{cost}(W_j) \geq \frac{n}{2} \cdot \left(\frac{B}{2} - \varepsilon\right)$. Moreover, as lottery $\{(\lambda_j, W_j)\}_{j\in[s]}$ is an implementation of the feasible fractional outcome $\vec{p}$, we have

$$B = \sum_{j\in[s]} \lambda_j \cdot \text{cost}(W_j) \geq \sum_{j\in[s]} \lambda_j \cdot \frac{n}{2} \cdot \left(\frac{B}{2} - \varepsilon\right),$$

which implies $\sum_{j\in[s]} \lambda_j \leq \frac{2}{n} \cdot \frac{B}{B/2-\varepsilon} < 1$, contradicting the assumption that $\{(\lambda_j, W_j)\}_{j\in[s]}$ is an implementation of $\vec{p}$. $\square$

Feasibility of $\vec{p}$ means that $B = \sum_{c\in C} p_c \cdot \text{cost}(c) = \sum_{c\in C\setminus\{g^*\}} p_c \cdot \left(\frac{B}{2} - \varepsilon\right) + p_{g^*} \cdot \frac{B}{2}$. Thus,

$$\sum_{c\in C} p_c = \frac{B - B/2 \cdot p_{g^*}}{B/2 - \varepsilon} + p_{g^*} = \frac{B - \varepsilon \cdot p_{g^*}}{B/2 - \varepsilon}.$$

Algorithm 2: BW-GCR-PB: Strong UFS and FJR

Input: Voters $N = [n]$, projects $C = [m]$, cost function cost, budget B, and utilities $(\vec{u}_i)_{i \in N}$.

$W_{\text{GCR}} \leftarrow \text{GCR}(N, C, \text{cost}, B, (u_i)_{i \in N})$

$\vec{p} \leftarrow \vec{1}_{W_{\text{GCR}}}$

$\widetilde{N} \leftarrow \emptyset$

$b_i \leftarrow 0$ for all $i \in N$

Let $\{N^1, \dots, N^\eta\}$ be the unanimous groups of N.

foreach $z \in [\eta]$ **do**

  if $|A_{N^z} \cap W_{GCR}| = |G_{N^z}|$ **then**

    $\widetilde{N} \leftarrow \widetilde{N} \cup N^z$

    $b_i \leftarrow \frac{B}{n} - \frac{1}{|N^z|} \cdot \text{cost}(G_{N^z})$ for all $i \in N^z$

    Let voters N^z spend their total budget $|N^z| \cdot \frac{B}{n} - \text{cost}(G_{N^z})$ on project $c \in A_{N^z}$ with the smallest cost, provided the updated $p_c \leq 1$.

Increase $\vec{p}$ arbitrarily such that for all $c \in C$, $p_c \leq 1$ and $\text{cost}(\vec{p}) = B$.

Obtain an outcome W sampled from the lottery implementing $\vec{p}$ by applying Theorem 3.2.

return $\vec{p}$ *and* W

Since $\vec{p}$ satisfies GFS with respect to voters N, we thus have

$$\frac{B - \varepsilon \cdot p_{g^*}}{B/2 - \varepsilon} = \sum_{c \in C} p_c \geq \frac{1}{n} \cdot \sum_{i \in N} \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}) = \frac{1}{n} \cdot \sum_{i \in N} \frac{B}{B/2 - \varepsilon} = \frac{B}{B/2 - \varepsilon},$$

a contradiction because $p_{g^*} > 0$. □

As demonstrated in Section 4.1, there is no logical dependence between GFS and Strong UFS in the approval-based committee voting. It is thus unclear whether ex-ante Strong UFS can be compatible with any ex-post fairness properties. We answer the question in the affirmative below.

6.2 Ex-ante Strong UFS + Ex-post FJR

We now show that if we only focus on giving ex-ante fair share guarantees to *unanimous* (instead of any) groups, ex-ante Strong UFS is compatible even with ex-post FJR.

Theorem 6.3. *In PB with binary utilities, Algorithm 2 computes an integral outcome sampled from a lottery that is ex-ante Strong UFS, ex-post BB1 and ex-post FJR.*

6.2.1 The Algorithm: BW-GCR-PB

Our algorithm, whose pseudocode can be found in Algorithm 2, starts by feeding the given PB instance into the *Greedy Cohesive Rule (GCR)* of Peters et al. [2021] and obtains an FJR outcome. More specifically, GCR begins by marking all voters as *active* and initializing $W = \emptyset$. In each step, GCR searches for a set of voters $N' \subseteq N$ who are all active and a set of projects $T \subseteq C \setminus W$ such that N' is weakly (β, T)-cohesive, breaking ties in favour of larger β, next smaller $\text{cost}(T)$, and then larger $|N'|$. GCR then includes projects T to W and labels voters N' as inactive. If, at any step, no weakly (β, T)-cohesive group exists for any positive integer β, then GCR returns W and terminates. Denote by r the number of steps GCR executes before terminating. For each $j \in [r]$, we refer to β_j, T_j and N_j as the values of β, T and N' for the weakly cohesive group selected in the j-th step of GCR. Denote by $W_{\text{GCR}} := \bigcup_{j \in [r]} T_j$ and initialize $\vec{p}$ as $\vec{1}_{W_{\text{GCR}}}$.

Algorithm 2 next loops over unanimous groups and set budgets for the voters. We first introduce additional notation for each unanimous group. Fix any unanimous group $S \subseteq N$. We denote by A_S the approval set of the unanimous group S (i.e., for all $i \in S$, $A_i = A_S$). Let us rename the projects in A_S in non-decreasing order of cost with arbitrary tie-breaking, i.e., $\text{cost}(g_1) \le \text{cost}(g_2) \le \cdots \le \text{cost}(g_{|A_S|})$. Denote by $G_S := \{g_1, g_2, \ldots, g_{\kappa_S}\}$ the maximal set of projects such that $\text{cost}(G_S) \le |S| \cdot \frac{B}{n}$. Put differently, if $A_S \setminus G_S \neq \emptyset$, $\text{cost}(G_S \cup \{g_{\kappa_S+1}\}) > |S| \cdot \frac{B}{n}$.

For ease of expression, let $\{N^1, N^2, \ldots, N^\eta\}$ be the partition of the *(maximal) unanimous groups* of voters N, i.e., for each $z \in [\eta]$, voters N^z are unanimous and for any $i \in N^z$ and $i' \in N^{z'}$ with $z \neq z'$, $A_i \neq A_{i'}$. Fix any $z \in [\eta]$. If voter $i \in N^z$ gets utility exactly $|G_{N^z}|$ from W_{GCR}, i.e., the `if`-condition holds, we set budget $b_i := \frac{B}{n} - \frac{1}{|N^z|} \cdot \text{cost}(G_{N^z})$. The unanimous group of voters N^z then spend their total budget of $|N^z| \cdot \frac{B}{n} - \text{cost}(G_{N^z})$ on project $c \in A_{N^z}$ with the smallest cost, provided the updated $p_c \le 1$. We will show shortly that the budget set-up is valid (Lemma 6.4) and guarantee that each unanimous group satisfies Strong UFS (Lemma 6.6). Line 11 then increases $\vec{p}$ in an arbitrary way so that $\vec{p}$ is feasible, that is, for all $c \in C$, $p_c \le 1$ and $\text{cost}(\vec{p}) = B$.

Finally, given the feasible fractional outcome $\vec{p}$, we apply the randomized rounding scheme (Theorem 3.2) and sample an outcome from the lottery implementing $\vec{p}$.

6.2.2 The Analysis of BW-GCR-PB

We begin by stating two key lemmas in the proof of Theorem 6.3. First, we prove that the total budgets we give the voters in line 9 is upper bounded by the leftover budget limit after selecting W_{GCR}.

Lemma 6.4. *$\sum_{i \in N} b_i \le B - \text{cost}(W_{GCR})$.*

Proof. For ease of exposition, in this proof, we *re-order* the r weakly cohesive groups encountered by GCR in line 1. Let $r' \in \{0, 1, 2, \ldots, r\}$ be an index such that for all $j \in [r']$, $N_j \cap \widetilde{N} \neq \emptyset$, i.e., there exists a unanimous group in N_j such that the `if`-condition holds. For each $j \in [r]$, let $\{N_j^1, N_j^2, \ldots, N_j^{\eta_j}\}$ be the partition of the (maximal) unanimous groups of voters N_j. We also assume without loss of generality that the first $\eta'_j \in \{0, 1, 2, \ldots, \eta_j\}$ unanimous groups are the ones such that the `if`-condition holds. Note that $\eta'_j \ge 1$ for all $j \in [r']$. Denote by $N_{\text{GCR}} := \bigcup_{j \in [r]} N_j$ the set of inactive voters due to GCR. Note that for all $i \in N$, $b_i \le \frac{B}{n}$. We will also make use of the following claim:

Claim 6.5. $\forall\, j \in [r'], z \in [\eta'_j], \text{cost}(T_j) \le \text{cost}(G_{N_j^z})$.

Proof. For ease of expression, let $T_{[j-1]} := \bigcup_{t=1}^{j-1} T_t$ denote the set of projects added by GCR in the first $j-1$ steps and $G'_{N_j^z} := G_{N_j^z} \setminus T_{[j-1]}$. Suppose for the sake of contradiction that $\text{cost}(G_{N_j^z}) < \text{cost}(T_j)$. Recall that the unanimous group N_j^z is $G_{N_j^z}$-cohesive. It follows that N_j^z is $G'_{N_j^z}$-cohesive because $|N_j^z| \cdot \frac{B}{n} \ge \text{cost}(G_{N_j^z}) \ge \text{cost}(G'_{N_j^z})$ and $G'_{N_j^z} \subseteq A_{N_j^z}$.

We first show that $\beta_j = |G'_{N_j^z}|$. If $\beta_j < |G'_{N_j^z}|$, then the j-th step of GCR would have added projects $G'_{N_j^z}$ (instead of T_j) because N_j^z is $G'_{N_j^z}$-cohesive and GCR breaks ties in favor of larger β. If $\beta_j > |G'_{N_j^z}|$, then $\beta_j \ge |G'_{N_j^z}| + 1$. Recall that $j \in [r']$ and $z \in [\eta'_j]$, meaning $|A_{N_j^z} \cap W_{\text{GCR}}| = |G_{N_j^z}|$.

We, however, have

$$\begin{aligned}
|G_{N_j^z}| &= |A_{N_j^z} \cap W_{\text{GCR}}| \\
&\geq |A_{N_j^z} \cap T_{[j]}| = |A_{N_j^z} \cap T_{[j-1]}| + |A_{N_j^z} \cap T_j| \\
&= |A_{N_j^z} \cap T_{[j-1]}| + \beta_j \\
&\geq |A_{N_j^z} \cap T_{[j-1]}| + |G'_{N_j^z}| + 1 \\
&\geq |G_{N_j^z} \cap T_{[j-1]}| + |G_{N_j^z} \setminus T_{[j-1]}| + 1 \\
&= |G_{N_j^z}| + 1,
\end{aligned}$$

a contradiction.

However, if $\beta_j = |G'_{N_j^z}|$, then the j-th step of GCR would have added projects $G'_{N_j^z}$. This is because N_j^z is $G'_{N_j^z}$-cohesive and conditioning on the same β value, GCR breaks ties in favor of $G'_{N_j^z}$ due to $\text{cost}(G'_{N_j^z}) \leq \text{cost}(G_{N_j^z}) < \text{cost}(T_j)$. □

We are now ready to establish the statement of the lemma:

$$\begin{aligned}
\sum_{i \in N} b_i &= \sum_{i \in N_{\text{GCR}}} b_i + \sum_{i \in N \setminus N_{\text{GCR}}} b_i \\
&= \sum_{j=1}^{r'} \sum_{z=1}^{\eta'_j} \left(|N_j^z| \cdot \frac{B}{n} - \text{cost}(G_{N_j^z}) \right) + \sum_{i \in N \setminus N_{\text{GCR}}} b_i \\
&\leq \sum_{j=1}^{r'} \left(|N_j| \cdot \frac{B}{n} - \sum_{z=1}^{\eta'_j} \text{cost}(G_{N_j^z}) \right) + \sum_{i \in N \setminus N_{\text{GCR}}} b_i \\
&\leq \sum_{j=1}^{r'} \left(|N_j| \cdot \frac{B}{n} - \text{cost}(T_j) \right) + \sum_{i \in N \setminus N_{\text{GCR}}} b_i \\
&\leq \sum_{j=1}^{r} \left(|N_j| \cdot \frac{B}{n} - \text{cost}(T_j) \right) + \sum_{i \in N \setminus N_{\text{GCR}}} b_i \\
&\leq |N_{\text{GCR}}| \cdot \frac{B}{n} - \text{cost}(W_{\text{GCR}}) + |N \setminus N_{\text{GCR}}| \cdot \frac{B}{n} \\
&= B - \text{cost}(W_{\text{GCR}}),
\end{aligned}$$

where the fourth transition is due to Claim 6.5 and $\eta'_j \geq 1$, and the fifth transition is due to weak cohesiveness. □

Our next result establishes that $\vec{p}$ satisfies Strong UFS.

Lemma 6.6. *Algorithm 2 outputs a fractional outcome $\vec{p}$ that satisfies Strong UFS.*

Proof. Let us first establish connections between unanimous groups considered by Strong UFS and cohesive groups considered by EJR.[10] Recall that $G_S := \{g_1, g_2, \ldots, g_{\kappa_S}\}$ is the maximal subset of projects approved by unanimous group S (in non-decreasing order of cost) such that $\text{cost}(G_S) \leq |S| \cdot \frac{B}{n}$. Since $|S| \cdot \frac{B}{n} \geq \text{cost}(G_S)$ and $G_S \subseteq A_S$, we observe that the unanimous group S is in fact G_S-cohesive.

[10]Since FJR implies EJR, our discussion is carried over to weakly cohesive groups considered by FJR.

It follows that given an EJR (or FJR) outcome W, for *all* $i \in S$, $|A_i \cap W| \geq |G_S|$. We thus conclude:

Claim 6.7. *Given an instance of PB with binary utilities, fix any unanimous group of voters $S \subseteq N$ and any EJR (or FJR) outcome W, then, for all $i \in S$, $|A_i \cap W| \geq |G_S|$.*

We now provide an alternative description of Equation (2) in the definition of Strong UFS. According to the RHS of Equation (2), the group S is endowed with a budget of $|S| \cdot \frac{B}{n}$ to select a fractional outcome. An optimal fractional outcome can be achieved by fully funding G_S and next funding a δ_S fraction of project g_{κ_S+1}, where $\delta_S := \frac{|S|\cdot \frac{B}{n} - \mathrm{cost}(G_S)}{\mathrm{cost}(g_{\kappa_S+1})} < 1$. Thus, in this case, we can rewrite the RHS of Equation (2):

$$\max_{\vec{t} \in \mathcal{X}\left(|S|\cdot\frac{B}{n}\right)} u_i(\vec{t}) = |G_S| + \delta_S. \tag{3}$$

We are now prepared to show that each unanimous group is satisfied with respect to Strong UFS. Fix any $z \in [\eta]$ and any voter $i \in N^z$. Since W_{GCR} satisfies FJR, by Claim 6.7, $|A_i \cap W_{\text{GCR}}| \geq |G_{N^z}|$. If $A_{N^z} = G_{N^z}$, then $u_i(\vec{p}) \geq |A_i \cap W_{\text{GCR}}| \geq |A_{N^z}|$, meaning that voter i is satisfied with respect to Strong UFS. We thus assume from now on $A_{N^z} \setminus G_{N^z} \neq \emptyset$. Put differently, the project $g_{\kappa_{N^z}+1}$ is well-defined. We will use the RHS of Equation (3) as the target utility to reason about Strong UFS. Specifically, we will show that $u_i(\vec{p}) \geq |G_{N^z}| + \delta_{N^z}$. We will mainly distinguish cases between $|A_i \cap W_{\text{GCR}}| \geq |G_{N^z}| + 1$ and $|A_i \cap W_{\text{GCR}}| = |G_{N^z}|$.

If $|A_i \cap W_{\text{GCR}}| \geq |G_{N^z}| + 1$, Strong UFS is satisfied as

$$u_i(\vec{p}) \geq |A_i \cap W_{\text{GCR}}| \geq |G_{N^z}| + 1 > |G_{N^z}| + \delta_{N^z} = \max_{\vec{t} \in \mathcal{X}\left(|N^z|\cdot\frac{B}{n}\right)} u_i(\vec{t}).$$

We now move on to the case where $|A_i \cap W_{\text{GCR}}| = |G_{N^z}|$. If $A_{N^z} \subseteq W_{\text{GCR}}$, clearly, Strong UFS is already satisfied. We thus assume $A_{N^z} \setminus W_{\text{GCR}} \neq \emptyset$. It means that there must exist a project $g \in \{g_1, g_2, \ldots, g_{\kappa_{N^z}}, g_{\kappa_{N^z}+1}\}$ such that $g \notin W_{\text{GCR}}$ and $\mathrm{cost}(g) \leq g_{\kappa_{N^z}+1}$. Line 9 of Algorithm 2 gives voters N^z a total budget of $|N^z| \cdot \frac{B}{n} - \mathrm{cost}(G_{N^z})$ to spend on project g. It may be that project g has already been funded to some extent in previous step(s), but this only helps voter N^z accumulate higher utilities. It follows easily that

$$\begin{aligned} u_i(\vec{p}) &\geq |A_i \cap W_{\text{GCR}}| + \frac{|N^z| \cdot \frac{B}{n} - \mathrm{cost}(G_{N^z})}{\mathrm{cost}(g)} \\ &\geq |G_{N^z}| + \frac{|N^z| \cdot \frac{B}{n} - \mathrm{cost}(G_{N^z})}{\mathrm{cost}(g_{\kappa_{N^z}+1})} = |G_{N^z}| + \delta_{N^z} = \max_{\vec{t} \in \mathcal{X}\left(|N^z|\cdot\frac{B}{n}\right)} u_i(\vec{t}). \end{aligned}$$ □

Using these statements, we now prove Theorem 6.3.

Proof of Theorem 6.3. We first show that the fractional outcome $\vec{p}$ returned by Algorithm 2 is feasible, i.e., for all $c \in C$, $p_c \leq 1$ and $\mathrm{cost}(\vec{p}) = B$. First, by the design of Algorithm 2, we maintain $p_c \leq 1$ for all $c \in C$ at any step. Next, we show that $\mathrm{cost}(\vec{p}) = B$. By Lemma 6.4, we have $\mathrm{cost}(W_{\text{GCR}}) + \sum_{i \in N} b_i \leq B$, meaning that $\mathrm{cost}(\vec{p}) \leq B$ before line 11 executes. Finally, according to how we increase $\vec{p}$ in line 11 and our assumption that $\mathrm{cost}(C) \geq B$, we conclude that the final fractional outcome $\vec{p}$ is feasible.

Next, by Lemma 6.6, the fractional outcome $\vec{p}$ returned by Algorithm 2 satisfies Strong UFS.

Finally, by Theorem 3.2, the lottery which implements $\vec{p}$ thus satisfies ex-ante Strong UFS as well as the sampled outcome W satisfies ex-post BB1. Ex-post FJR follows from Peters et al. [2021] (W_{GCR} is FJR) and from Theorem 3.2 that W_{GCR} is included in the integral outcome sampled. □

Algorithm 3: BW-MES-PB: Strong UFS and EJR

Input: Voters $N = [n]$, projects $C = [m]$, budget B, cost function cost, and utilities $(\vec{u}_i)_{i \in N}$.
$W_{\text{MES}} \leftarrow \text{MES}(N, C, \text{cost}, B, (u_i)_{i \in N})$
$\vec{p} \leftarrow \vec{1}_{W_{\text{MES}}}$
Let y_{ij} for each $i \in N$ and $j \in W_{\text{MES}}$ be the amount voter i spent on project j during MES.
$b_i \leftarrow \frac{B}{n} - \sum_{j \in W_{\text{MES}}} y_{ij}$ for all $i \in N$, which is the *remaining* budget of voter i after MES
$N' \leftarrow \{i \in N \mid A_i \setminus W_{\text{MES}} \neq \emptyset\}$
foreach $i \in N'$ **do**
    Let $\kappa_i \in \arg\min_{c \in A_i \setminus W_{\text{MES}}} \text{cost}(c)$
    $y_{i\kappa_i} \leftarrow b_i$
foreach $i \in N \setminus N'$ **do**
    Voter i spends b_i arbitrarily provided $\sum_{i \in N} y_{ij} \leq \text{cost}(j)$ for all $j \in C$.
foreach $j \in C$ **do** $p_j \leftarrow \frac{\sum_{i \in N} y_{ij}}{\text{cost}(j)}$.
Obtain an outcome W sampled from the lottery implementing $\vec{p}$ by applying Theorem 3.2.
return $\vec{p}$ *and* W

6.3 Ex-ante Strong UFS + Ex-post EJR (in Polynomial Time)

Despite providing strong ex-ante and ex-post fairness guarantees, BW-GCR-PB (Algorithm 2) does not run in polynomial time. We present here a polynomial-time algorithm that is ex-ante Strong UFS, at the cost of weakening ex-post fairness guarantee to EJR.

Theorem 6.8. *In PB with binary utilities, Algorithm 3 computes an integral outcome sampled from a lottery that is ex-ante Strong UFS, ex-post BB1, and ex-post EJR in polynomial time.*

At a high level, our algorithm BW-MES-PB (Algorithm 3) gives each voter an initial budget of B/n and uses the *Method of Equal Shares (MES)* of Peters et al. [2021] as a subroutine to obtain an EJR outcome W_{MES}. We now describe MES and its necessary components. Each voter is initially given a budget of B/n. We start with $W = \emptyset$ and sequentially add projects to W. For each selected project $j \in W$, we write y_{ij} for the amount that voter i pays for j; we require that $\sum_{i \in N} y_{ij} = \text{cost}(j)$. We write $b_i = B/n - \sum_{j \in W} y_{ij} \geq 0$ for the amount of budget voter i has left. For $\rho \geq 0$, we say that a project $j \notin W$ is *ρ-affordable* if

$$\sum_{i \in N} \min(b_i, u_i(j) \cdot \rho) = \text{cost}(j).$$

If no project is ρ-affordable for any ρ, MES terminates and returns W. Otherwise, it selects the project $j \in C \setminus W$ that is ρ-affordable for minimum ρ. Payments are given by $y_{ij} = \min(b_i, u_i(j) \cdot \rho)$.

A key step in the proof of Theorem 6.8 is to show that for each unanimous group $N^z \subseteq N$, the remaining budget of the group $\sum_{i \in N^z} \left(\frac{B}{n} - \sum_{c \in W_{\text{MES}}} y_{ic}\right)$ is at least $|N^z| \cdot \frac{B}{n} - \text{cost}(G_{N^z})$. As a result, the group together can use their remaining budget to fund the project with the smallest cost and be satisfied with respect to Strong UFS. We now prove the theorem.

Proof of Theorem 6.8. First, we show that the fractional outcome $\vec{p}$ returned by Algorithm 3 is feasible, i.e., $\text{cost}(\vec{p}) = B$. By the design of Algorithm 3, at any step, we maintain $p_c \leq 1$ for all $c \in C$. Next, since each voter starts with a budget of B/n and spends the entirety of their budget, by the construction of $\vec{p}$ in line 11 we have that

$$\sum_{j \in C} p_j \cdot \mathrm{cost}(j) = \sum_{j \in C} \frac{\sum_{i \in N} y_{ij}}{\mathrm{cost}(j)} \cdot \mathrm{cost}(j) = \sum_{i \in N} \sum_{j \in C} y_{ij} = \sum_{i \in N} B/n = B.$$

Next, we show that the fractional outcome $\vec{p}$ satisfies Strong UFS. The proof idea is similar to that of Lemma 6.6. Fix any $z \in \eta$ and any voter $i \in N^z$. If $A_i \subseteq W_{\mathrm{MES}}$, then voter i already gets the highest possible utility and thus is satisfied with respect to Strong UFS. We hence assume from now on that $A_i \setminus W_{\mathrm{MES}} \neq \emptyset$.

Since W_{MES} satisfies EJR, by Claim 6.7, $|A_i \cap W_{\mathrm{MES}}| \geq |G_{N^z}|$. If $A_{N^z} = G_{N^z}$, then

$$u_i(\vec{p}) \geq |A_i \cap W_{\mathrm{MES}}| \geq |G_{N^z}| = |A_{N^z}|,$$

implying that Strong UFS is satisfied. We thus focus on the case where $A_{N^z} \setminus G_{N^z} \neq \emptyset$. In other words, the project $g_{\kappa_{N^z}+1}$ is well-defined. From Equation (3), it suffices for us to show $u_i(\vec{p}) \geq |G_{N^z}| + \delta_{N^z}$. If $|A_i \cap W_{\mathrm{MES}}| \geq |G_{N^z}| + 1$, clearly, Strong UFS is satisfied. We now consider the case where $|A_i \cap W_{\mathrm{MES}}| = |G_{N^z}|$.

It can be observed from the definition of MES that for any unanimous group $S \subseteq N$ and for any pair of voters $i, j \in S$, $b_i = b_j$ at any step; moreover, for any project $c \in C$, $y_{ic} = y_{jc}$. We show below that

$$|N^z| \cdot \sum_{c \in A_{N^z} \cap W_{\mathrm{MES}}} y_{ic} \leq \mathrm{cost}(G_{N^z}). \tag{4}$$

Put differently, the payments the unanimous group N^z made during MES is at most the amount needed to buy G_{N^z}. Recall that $G_{N^z} = \{g_1, g_2, \ldots, g_{\kappa_{N^z}}\}$ is the best set of projects voters N^z can afford as a group. Let $\{h_1, h_2, \ldots, h_t\} \subseteq A_i \cap W_{\mathrm{MES}}$ denote the first t projects added by MES from $A_i \cap W_{\mathrm{MES}}$. More specifically, we show that for each $t = 1, 2, \ldots, |G_{N^z}|$,

$$|N^z| \cdot y_{ih_t} \leq \mathrm{cost}(g_t).$$

Suppose for the sake of contradiction that $|N^z| \cdot y_{ih_t} > \mathrm{cost}(g_t)$. By the definition of being ρ-affordable, at the step where MES includes project h_t, the ρ-value is at least y_{ih_t}. Note that at the moment, there still exists some project $g \in \{g_1, g_2, \ldots, g_t\}$ available to be funded, and, $\mathrm{cost}(g) \leq \mathrm{cost}(g_t)$. As a result, project g is ρ-affordable with ρ-value upper bounded by $\frac{\mathrm{cost}(g_t)}{|N^z|}$, which is less than y_{ih_t}, a contradiction.

Let $h \in \arg\min_{c \in A_i \setminus W_{\mathrm{MES}}} \mathrm{cost}(c)$. Since $|A_i \cap W_{\mathrm{MES}}| = |G_{N^z}|$, we have $h \in \{g_1, g_2, \ldots, g_{\kappa_{N^z}}, g_{\kappa_{N^z}+1}\}$ and $\mathrm{cost}(h) \leq \mathrm{cost}(g_{\kappa_{N^z}+1})$. As a result, we are able to show that Strong UFS is satisfied:

$$\begin{aligned}
\sum_{c \in A_i} p_c &= \sum_{c \in A_i} \frac{\sum_{j \in N} y_{jc}}{\mathrm{cost}(c)} \\
&= \sum_{c \in A_i \cap W_{\mathrm{MES}}} \frac{\sum_{j \in N} y_{jc}}{\mathrm{cost}(c)} + \sum_{c \in A_i \setminus W_{\mathrm{MES}}} \frac{\sum_{j \in N} y_{jc}}{\mathrm{cost}(c)} \\
&= |A_i \cap W_{\mathrm{MES}}| + \sum_{c \in A_i \setminus W_{\mathrm{MES}}} \frac{\sum_{j \in N} y_{jc}}{\mathrm{cost}(c)} \\
&\geq |A_i \cap W_{\mathrm{MES}}| + \frac{\sum_{j \in N} y_{jh}}{\mathrm{cost}(h)} \\
&\geq |A_i \cap W_{\mathrm{MES}}| + \frac{\sum_{j \in N^z} y_{jh}}{\mathrm{cost}(h)}
\end{aligned}$$

$$
\begin{aligned}
&\geq |A_i \cap W_{\text{MES}}| + \frac{|N^z| \cdot y_{ih}}{\text{cost}(h)} \\
&= |A_i \cap W_{\text{MES}}| + \frac{|N^z| \cdot \left(\frac{B}{n} - \sum_{c \in A_i \cap W_{\text{MES}}} y_{ic}\right)}{\text{cost}(h)} \\
&\geq |A_i \cap W_{\text{MES}}| + \frac{|N^z| \cdot \frac{B}{n} - \text{cost}(G_{N^z})}{\text{cost}(h)} \quad (\because \text{Equation (4)}) \\
&\geq |A_i \cap W_{\text{MES}}| + \frac{|N^z| \cdot \frac{B}{n} - \text{cost}(G_{N^z})}{\text{cost}(g_{\kappa_{N^z}+1})} \\
&= |G_{N^z}| + \delta_{N^z} = \max_{\vec{t} \in \mathcal{X}\left(|N^z| \cdot \frac{B}{n}\right)} u_i(\vec{t}).
\end{aligned}
$$

Finally, by Theorem 3.2, the lottery which implements $\vec{p}$ satisfies ex-ante Strong UFS and the sampled outcome W satisfies ex-post BB1. Ex-post EJR follows from Peters et al. [2021] (W_{MES} is EJR) and from Theorem 3.2 that W_{MES} is included in the integral outcome sampled. □

7 BoBW Fairness in General PB

We now move on to the setting of general PB, in which we show a strong impossibility that ex-ante IFS and ex-post JR are not compatible, even in the unit-cost PB setting. This impossibility is striking as, even in the general setting, the much stronger properties of ex-ante GFS and ex-post FJR are independently achievable via Fractional Random Dictator (Theorem 4.13) and Greedy Cohesive Rule (GCR) [Peters et al., 2021], respectively.

In what follows, we show that ex-ante IFS and ex-post JR are incompatible in the unit-cost setting with cardinal utilities. Intuitively, in situations where voters have high utilities for distinct projects, the outcomes that guarantee the highest expected utility may not include a project which gives every voter non-zero utility.

Theorem 7.1. *In unit-cost PB, ex-ante IFS and ex-post JR are incompatible.*

Proof. Consider any instance of unit-cost PB with $n \geq 4$, $m = 2n + 1$, and in which $k = 2$. Suppose the project set is composed of (i) one project c, for which $u_{ic} = 1$ for all $i \in N$, and additionally (ii) a set of two projects G_i for each agent $i \in N$, such that agent i gets value H from each project in G_i and every other agent $i' \neq i$ gets zero utility from each project in G_i.

We begin by showing that any integral outcome which satisfies ex-post JR must contain c. Suppose, for a contradiction, that there exists some integral committee W which satisfies ex-post JR and which does not contain c. Since $c \notin W$, there are at least $n - 2$ agents who receive zero utility from W. Denote this group of agents S. Note that $n \geq 4 \implies n - 2 \geq n/2$, and each agent in S receives one utility from c. Thus, S are (α, c)-cohesive for any mapping α with $\alpha(j) = 1$. However, each agent in S receives zero utility from W which contradicts that W satisfies ex-post JR.

Thus, any randomized committee which satisfies ex-post JR for our instance must be of the form $\{(\lambda_1, \{c, w_1\}), \ldots, (\lambda_r, \{c, w_r\})$ where, for each $j \in [r]$, $w_j \in H_i$ for some $i \in N$. Denote by $\vec{p}$ any fractional committee which such a randomized committee implements. If we sum the LHS of the IFS guarantees of all agents in N, we have that

$$
\sum_{i \in N} u_i(\vec{p}) = \sum_{j \in [r]} \lambda_j \sum_{i \in N} (1 + u_{iw_j}) = \sum_{j \in [r]} \lambda_j (n + H) = n + H.
$$

Thus, for some agent $i \in N$, it holds that $u_i(\vec{p}) \leq 1 + H/n$. Finally, see that $\frac{1}{n} \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}) = 2H/n$. Together, these observations show us that for any instance in which $H > n$, there will be at least one which agent who does not receive their IFS guarantee. □

Since ex-ante IFS is incompatible with ex-post JR in even the restricted case with unit costs, it is natural to ask whether we can guarantee *any* non-trivial ex-ante individual representation in conjunction with ex-post fairness. In fact, even in the general PB setting, we can guarantee strictly positive expected utility to each voter — a property known as *positive share* — in conjunction with the much stronger ex-post property FJR. We will not prove this formally, but note that it follows from the observation that a result akin to Theorem 5.12 holds in the general PB setting if we replace GRP with ex-ante positive share. In more detail, in the general PB setting, affordability guarantees that there exist payment functions which cover the costs of projects and under which voters only pay for projects they receive non-zero utility from. Thus, we can "complete" any affordable integral outcome to a fractional outcome satisfying ex-ante positive share by spending voters' remaining budgets on any remaining projects they receive utility from. If no budget remains for some voter, then that voter's representation follows from affordability. By applying our usual implementation technique, we can additionally guarantee ex-post BB1, leading to the following statement.

Proposition 7.2. *In general PB, a lottery satisfying ex-ante positive share, ex-post BB1, and ex-post FJR always exists.*

This follows from the discussion in the previous paragraph because Peters et al. [2021] showed GCR always returns an integral outcome which satisfies affordability and ex-post FJR. Since GCR requires exponential time, it is worth noting that MES satisfies EJR up to one project (EJR-1) and affordability and runs in polynomial time [Peters et al., 2021]. Thus, an integral outcome sampled from a lottery that is ex-ante positive share, ex-post BB1, and ex-post EJR-1 can be computed in polynomial time.

Note that, whereas in the cost utilities case in Section 5.2, affordability was sufficient to guarantee an outcome could be completed to one which satisfies GRP (and thus Strong UFS and GFS), that is far from the case in the general utilities. This is because affordability does not capture fairness as effectively in the general setting: it does not require that voters' payments prioritize their favorite projects, something that is not necessary in the cost utilities case where voters are always indifferent about how they spend an additional unit. This observation also explains why it is not obvious how one could apply the flow network approach used in Section 5.2 to the general PB (or unit-cost PB) setting.

8 Conclusion

In this paper, we initiated the study of PB lotteries and used this approach to study best-of-both-worlds fairness in PB. We provided a complete set of results for several natural PB special cases (cf. Figure 2). Specifically, we gave algorithms which compute a lottery that guarantees each voter certain expected utility while maintaining the strongest indivisible PB fairness notions ex post.

In future research, it is an interesting direction to use *approximate* fair share notions to circumvent our impossibilities. More briefly, the proof of Theorem 7.1 shows something slightly stronger than the statement: in unit-cost PB, ex-post JR is incompatible with any α-approximation of IFS for $\alpha < 2$. We complemented this result by showing that positive share remains compatible with much stronger ex-post properties, such as FJR. A more detailed analysis of GCR and MES may show that FJR is in fact compatible with a constant-factor approximation to IFS. Moreover, while we focused on fairness, it is an interesting future direction to seek best-of-both-worlds results for other desiderata, such as economic efficiency.

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A Relations Between Ex-ante Fairness Notions in Approval-Based Committee Voting (Cont.)

We continue our discussions in Section 4.1 regarding the relations between the proposed ex-ante fairness notions in the setting of ABC voting. First of all, recall from Footnote 6 that Random Dictator satisfies GFS but not Strong IFS. Therefore, GFS (and thus UFS) does not imply Strong IFS, let alone Strong UFS or cake EJR.

Next, Strong IFS does not imply UFS, which can be seen from the following example.

Example A.1 (Strong IFS does not imply UFS)**.** *Consider an instance with $n = 3$, $k = 2$, and approval preferences*

$$A_1 = A_2 = \{c_1, c_2\} \qquad A_3 = \{c_3, c_4\}.$$

Observe that the fractional committee $\vec{p} = \left(\frac{2}{3}, 0, 1, \frac{1}{3}\right)$ satisfies Strong IFS. However, $\vec{p}$ does not satisfy UFS with respect to the group $S = \{1, 2\}$ since

$$\frac{2}{3} = \sum_{c \in \{c_1, c_2\}} p_c < \frac{|S|}{n} \cdot \min\{k, |A_i|\} = \frac{4}{3}.$$

We now show cake EJR (and thus Strong UFS or Strong IFS) does not imply GFS, let alone GRP.

Example A.2 (Cake EJR does not imply GFS)**.** *Consider an instance with $n = 3$, $k = 2$, and the following approval preferences:*

$$A_1 = \{c_1, c_2\} \qquad A_2 = \{c_1, c_3\} \qquad A_3 = \{c_4, c_5\}.$$

Observe that the fractional committee $\vec{p} = (1, 0, 0, 1, 0)$ satisfies cake EJR:

- *The 2-voter group $\{1, 2\}$ commonly approves a single candidate, is large enough (i.e., $2 = |\{1, 2\}| \geq t \cdot \frac{n}{k} = 1 \times \frac{3}{2}$), and a voter in the group (say voter 1) gets a utility of 1 from $\vec{p}$.*
- *The 1-voter group $\{3\}$ commonly approves two candidates. The group is large enough for $t \leq \frac{2}{3}$, i.e., $1 = |\{3\}| = t \cdot \frac{n}{k} = \frac{2}{3} \cdot \frac{3}{2}$. Voter 3 gets a utility of at least $\frac{2}{3}$ from $\vec{p}$, as desired.*

However, $\vec{p}$ does not satisfy GFS with respect to the group $S = \{1, 2\}$ since

$$1 = \sum_{c \in \bigcup_{i \in S} A_i} p_c < \sum_{i \in S} \frac{1}{n} \cdot \min\{k, |A_i|\} = \frac{4}{3}.$$

Finally, we show GRP does not imply cake EJR.

Example A.3 (GRP does not imply cake EJR)**.** *Consider again the instance used in Example A.2. We now use a different fractional committee to show that GRP does not imply cake EJR.*

Observe that the fractional committee $\vec{p} = (0, \frac{2}{3}, \frac{2}{3}, 0, \frac{2}{3})$ satisfies GRP. However, $\vec{p}$ does not satisfy cake EJR with respect to the group $S = \{1, 2\}$, because this 2-voter group commonly approves a single candidate (i.e., $t = 1$) and is large enough (i.e., $2 = |\{1, 2\}| \geq t \cdot \frac{n}{k} = 1 \times \frac{3}{2}$), but neither voter in the group gets a utility of at least 1.

B Omitted Proofs

B.1 Proof of Theorem 4.13

Let $\vec{p}$ be the fractional outcome returned by the Fractional Random Dictator algorithm. We first show that $\vec{p}$ is a feasible fractional outcome:

$$\begin{aligned}
\sum_{j \in C} p_j \cdot \mathrm{cost}(j) &= \frac{1}{n} \cdot \sum_{j \in C} \left(\mathrm{cost}(j) \cdot \sum_{i \in N} \left(\mathbb{1}_{\{j \in X_i\}} + \mathbb{1}_{\{j = g_i\}} \cdot \frac{B - \mathrm{cost}(X_i)}{\mathrm{cost}(j)} \right) \right) \\
&= \frac{1}{n} \cdot \sum_{i \in N} \sum_{j \in C} \left(\mathrm{cost}(j) \cdot \left(\mathbb{1}_{\{j \in X_i\}} + \mathbb{1}_{\{j = g_i\}} \frac{B - \mathrm{cost}(X_i)}{\mathrm{cost}(j)} \right) \right) \\
&= \frac{1}{n} \sum_{i \in N} \left(\mathrm{cost}(X_i) + \mathrm{cost}(g_i) \cdot \frac{B - \mathrm{cost}(X_i)}{\mathrm{cost}(g_i)} \right) = B.
\end{aligned}$$

Next, we show $\vec{p}$ satisfies ex-ante GFS. Fix any group of voters $S \subseteq N$, we have

$$\begin{aligned}
\sum_{j \in C} \left(p_j \cdot \max_{i \in S} u_i(j) \right) &= \frac{1}{n} \cdot \sum_{j \in C} \left(\max_{i \in S} u_i(j) \cdot \sum_{i \in N} \left(\mathbb{1}_{\{j \in X_i\}} + \mathbb{1}_{\{j = g_i\}} \cdot \frac{B - \mathrm{cost}(X_i)}{\mathrm{cost}(j)} \right) \right) \\
&\geq \frac{1}{n} \cdot \sum_{j \in C} \sum_{i \in S} \left(u_i(j) \cdot \mathbb{1}_{\{j \in X_i\}} + u_i(j) \cdot \mathbb{1}_{\{j = g_i\}} \cdot \frac{B - \mathrm{cost}(X_i)}{\mathrm{cost}(j)} \right) \\
&= \frac{1}{n} \cdot \sum_{i \in S} \left(\sum_{j \in X_i} u_i(j) + u_i(g_i) \cdot \frac{B - \mathrm{cost}(X_i)}{\mathrm{cost}(g_i)} \right) \\
&= \frac{1}{n} \cdot \sum_{i \in S} \max_{\vec{t} \in \mathcal{X}} u_i(\vec{t}).
\end{aligned}$$

The last transition follows since the quantity in brackets is exactly equal to the utility voter i receives from the fractional outcome they select in the Fractional Random Dictator algorithm, which is their optimal fractional outcome.

Lastly, it can be verified that the Fractional Random Dictator algorithm runs in polynomial time, as for each $i \in N$, computing X_i and g_i can be done in polynomial time.

B.2 Proof of Theorem 5.9

Given any instance of PB with cost utilities, let $\vec{p}$ be a fractional outcome of the instance satisfying GRP. We will show there exists a maximum flow f on the flow network $\mathcal{N}$ of the instance such that $p_c \cdot \mathrm{cost}(c) \geq f(c, t)$ for all $c \in C$. Let $\widetilde{\mathcal{N}}$ be a flow network only differing from $\mathcal{N}$ by the capacities on arcs entering the sink. Specifically, for $\widetilde{\mathcal{N}}$, let $\text{cap}(c,t) = p_c \cdot \text{cost}(c)$ for each $c \in C$. For some minimum s-t cut on the flow network $\widetilde{\mathcal{N}}$, let $T^* \subseteq N$ be the subset of voters in the same part of the cut as the source s. By the max-flow min-cut theorem, there exists a maximum flow f on $\widetilde{\mathcal{N}}$ such that

$$\begin{aligned}
\sum_{c \in C} f(c,t) &= |N \setminus T^*| \cdot \frac{B}{n} + \sum_{c \in \bigcup_{i \in T^*} A_i} p_c \cdot \text{cost}(c) \\
&\geq |N \setminus T^*| \cdot \frac{B}{n} + |T^*| \cdot \frac{B}{n} - \max_{T \subseteq T^*} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \\
&= B - \max_{T \subseteq T^*} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \\
&\geq B - \max_{T \subseteq N} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \\
&= B - \left[-\min_{T \subseteq N} - \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \right] \\
&= B + \min_{T \subseteq N} \left[-|T| \cdot \frac{B}{n} + \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right] \\
&= \min_{T \subseteq N} \left[|N \setminus T| \cdot \frac{B}{n} + \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right],
\end{aligned}$$

where the first inequality follows from the fact that $\vec{p}$ satisfies GRP. It is worth noting that the final expression above is the minimum cut value of the flow network $\mathcal{N}$, we can conclude (again by the max-flow min-cut theorem) that f is also a maximum flow on the flow network $\mathcal{N}$. Lastly, by feasibility on $\widetilde{\mathcal{N}}$, we have that $p_c \cdot \text{cost}(c) \geq f(c,t)$ for each $c \in C$.

We now proceed to prove the converse direction. Given any instance of PB with cost utilities, let $\vec{p}$ be a fractional outcome of the instance and f be a maximum flow on the network representation $\mathcal{N}$ such that $p_c \cdot \text{cost}(c) \geq f(c,t)$ for each $c \in C$. Suppose, on the contrary, that $\vec{p}$ does not satisfy GRP. That is, there exists $S \subseteq N$ such that

$$\sum_{c \in \bigcup_{i \in S} A_i} p_c < |S| \cdot \frac{B}{n} - \max_{T \subseteq S} \left[|T| \cdot \frac{B}{n} - \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right]. \tag{5}$$

Observe that the RHS of Equation (5) can be reformulated as $\min_{T \subseteq S} \left[|S \setminus T| \cdot \frac{B}{n} + \text{cost}(\bigcup_{i \in T} A_i) \right]$, which is the minimum cut value of a network representation $\mathcal{N}_S$ constrained on the set of voters S and the set of candidates $\bigcup_{i \in S} A_i$. By the max-flow min-cut theorem, we know that there exists a maximum flow f' on the flow network $\mathcal{N}_S$ whose value satisfies

$$\sum_{c \in \bigcup_{i \in S} A_i} f'(c,t) = \min_{T \subseteq S} \left[|S \setminus T| \cdot \frac{B}{n} + \text{cost}(\textstyle\bigcup_{i \in T} A_i) \right].$$

Construct a new flow f^* on the entire flow network $\mathcal{N}$ as follows:

- For all $i \in S$, $f^*(s,i) = f'(s,i)$, and $f^*(i,c) = f'(i,c)$ for all $c \in A_i$.
- For all $i \in N \setminus S$, $f^*(s,i) = f(s,i)$, $f^*(i,c) = 0$ for all $c \in \bigcup_{j \in S} A_j$, and $f^*(i,c) = f(i,c)$ for all $c \in A_i \setminus \bigcup_{j \in S} A_j$.
- For all $c \in \bigcup_{i \in S} A_i$, $f^*(c,t) = f'(c,t)$.

- For all $c \in C \setminus \bigcup_{i\in S} A_i$, $f^*(c,t) = f(c,t)$.

It can be verified easily that flow f^* satisfies both the capacity constraints and the flow conservation constraints on the flow network $\mathcal{N}$, and thus is a valid flow on $\mathcal{N}$. Finally we see that,

$$\begin{aligned}
\sum_{c\in C} f(c,t) &= \sum_{c\in\bigcup_{i\in S} A_i} f(c,t) + \sum_{c\in C\setminus\bigcup_{i\in S} A_i} f(c,t) \\
&\leq \sum_{c\in\bigcup_{i\in S} A_i} p_c \cdot \mathrm{cost}(c) + \sum_{c\in C\setminus\bigcup_{i\in S} A_i} f(c,t) \\
&< \left[|S| \cdot \frac{B}{n} - \max_{T\subseteq S} \left[|T| \cdot \frac{B}{n} - \mathrm{cost}(\textstyle\bigcup_{i\in T} A_i) \right] \right] + \sum_{c\in C\setminus\bigcup_{i\in S} A_i} f(c,t) && \because \text{Equation (5)} \\
&= \min_{T\subseteq S} \left[|S\setminus T| \cdot \frac{B}{n} + \mathrm{cost}(\textstyle\bigcup_{i\in T} A_i) \right] + \sum_{c\in C\setminus\bigcup_{i\in S} A_i} f(c,t) \\
&= \sum_{c\in\bigcup_{i\in S} A_i} f'(c,t) + \sum_{c\in C\setminus\bigcup_{i\in S} A_i} f(c,t) \\
&= \sum_{c\in C} f^*(c,t),
\end{aligned}$$

which contradicts the assumption that f is a maximum flow on $\mathcal{N}$.